Internat. J. Math. & Math. Sci.
VOL. 12 NO. 4 (1989) 713-720

SOLUTION MATCHING FOR BOUNDARY VALUE PROBLEMS FOR LINEAR EQUATIONS

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(Received March 17, 1988)

ABSTRACT. For the linear differential equation, $y^{(n)} = \sum_{i=1}^{n} a_i(x)y^{(i-1)}$, (1.1), where $n \geq 3$, solutions of multipoint boundary value problems on an interval $[a,c]$ are obtained, via the use of Liapunov-like functions, by matching solutions of certain boundary value problems for (1.1) on $[a,b]$ with solutions of other boundary value problems for (1.1) on $[b,c]$.

KEY WORDS AND PHRASES. Multipoint boundary value problem, solution matching, linear equation.
1980 MATHEMATICS SUBJECT CLASSIFICATION. 34B05, 34B10.

1. INTRODUCTION.

We will be concerned with the existence of solutions of k-point boundary value problems, $2 < k \leq n$, on an interval $[a,c]$ for the nth order linear differential equation

$$y^{(n)} = \sum_{i=1}^{n} a_i(x)y^{(i-1)}, \quad n \geq 3, \tag{1.1}$$

where the $a_i(x) \in C[a,c]$.

The point $b \in (a,c)$ will fixed throughout, and we will employ techniques which match a solution $y_1(x)$ of a $(k-1)$-point boundary value problem for (1.1) on $[a,b]$ with a solution $y_2(x)$ of a 2-point boundary value problem for (1.1) on $[b,c]$ such that, $y(x)$ defined by

$$y(x) = \begin{cases} y_1(x), & a \leq x \leq b, \\ y_2(x), & b \leq x \leq c, \end{cases}$$

is a solution of a k-point boundary value problem for (1.1) on $[a,c]$.

Solution matching techniques were first applied by Bailey, Shampine, and Waltman [1] where they dealt with solutions of 2-point boundary value problems for the second order equation $y'' = f(x, y, y')$ by matching solutions of initial value problems. Since then, a number of papers have appeared in which solutions of 3-point boundary value problems on $[a,c]$ for the nth order ordinary differential equation $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$ were obtained by matching solutions of 2-point problems on $[a,b]$ with 2-point problems on $[b,c]$; see, for example, Barr and Sherman [2], Das and Lalli [3], Henderson [4], Moorti and Garner [5], and Rao, Murthy and Rao [6]. In those papers [2 - 6], a monotonicity condition imposed on f plays an important role in the solution matching techniques in obtaining solutions of the 3-point problems. For example, in obtaining solutions of certain 3-point conjugate problems, Barr and Sherman [2] assumed that f satisfies the conditions:

(i) $u_{n-1} < v_{n-1}$, and $(-1)^{n-j}(u_j - v_j) \geq 0$, $j = 1, 2, \ldots, n-2$, imply $f(x, u_1, \ldots, u_{n-1}, w) < f(x, v_1, \ldots, v_{n-1}, w)$, for all $x \in (a,b]$ and all $w \in R$, and

(ii) $u_{n-1} < v_{n-1}$, and $u_j \leq v_j$, $j = 1, 2, \ldots, n-2$, imply $f(x, u_1, \ldots, u_{n-1}, w) < f(x, v_1, \ldots, v_{n-1}, w)$, for all $x \in [b,c)$ and all $w \in R$.

Moorti and Garner [5], by assuming (i) and (ii) with respect to third order equations, obtained solutions of certain 3-point focal problems by matching solutions. Das and Lalli [3] also assumed (i) and (ii) with respect to third order equations, (however in relaxing other assumptions which were made in [2], the proof of Theorem 2.1 in [3] does not appear to be valid). In the paper by Rao, Murthy and Rao [6], conditions (i) and (ii) were modified some and solution matching was applied to obtain solutions of 3-point conjugate problems for third order ordinary differential equations. Then Henderson [4] generalized the monotonicity conditions of [6] and used solution matching to obtain the existence of solutions of a large class of 3-point boundary value problems on $[a,c]$ for $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$.

However, to obtain solutions of k-point problems on $[a,c]$, for $3 < k \leq n$, by matching a solution $y_1(x)$ on $[a,b]$ with a solution $y_2(x)$ on $[b,c]$ under assumptions (i) and (ii) or the more general monotonicity conditions in [4], one cannot necessarily conclude that $y_1^{(n-2)}(b) = y_2^{(n-2)}(b)$, hence cannot conclude a solution on $[a,c]$. For the linear equation (1.1), using Liapunov-like functions, Barr and Miletta [7]matched solutions of $(n-1)$-point boundary value problems with solutions of 2-point boundary value problems, thus obtaining solutions of n-point boundary value problems. In this paper, using techniques similar to those of Barr and Miletta [7], we obtain solutions of k-point boundary value problems for (1.1) on $[a,c]$. In particular, given $2 < k \leq n$, positive integers $m_1,\ldots,m_k$ such that $\sum_{i=1}^{k} m_i = n$ and $m_k = 1$, points $a \leq x_1 < \ldots < x_{k-1} = b < x_k \leq c$, and $y_{ij} \in R$, $0 \leq i \leq m_j - 1$, $1 \leq j \leq k$, we are interested in solutions of boundary value problems for (1.1) satisfying

$$y^{(i)}(x_j) = y_{ij},\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ y(x_k) = y_{0,k}, \tag{1.2}$$

and for $\mu \in \{0,1,\dots,n-2\}$ fixed,

$$y^{(i)}(x_j) = y_{ij},\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ y^{(\mu)}(x_k) = y_{0,k}. \tag{1.3}$$

The existence of unique solutions of certain (k-1)-point and 2-point boundary value problems for (1.1) used in the matching to obtain solutions of(1.1), (1.2) and (1.1),(1.3) will be established through the use of Liapunov-like or control functions.

DEFINITIONS. Given $M \ge 0$ and $[\alpha,\beta] \subseteq [a,c]$, a control function

$$V_M(x,y_1,\dots,y_n):\ [\alpha,\beta] \times \mathbb{R}^n \to \mathbb{R}$$

is a function which is continuous, locally Lipschitz with respect to $(y_1,\dots,y_n)$ and satisfies

$$\text{i) } V_M(x,y_1,\dots,y_n) = 0, \text{ if } y_{n-1} = M,$$

$$\text{ii) } V_M(x,y_1,\dots,y_n) > 0, \text{ if } y_{n-1} > M.$$

Corresponding to $V_M(x,y_1,\dots,y_n)$ and a solution $y(x)$ of the differential equation (1.1), define

$$V_M'(x,y(x),\dots,y^{(n-1)}(x)) = \liminf_{h \to 0+} \frac{1}{h}\left[V_M(x+h,y(x+h),\dots,y^{(n-1)}(x+h)) - V_M(x,y(x),\dots,y^{(n-1)}(x))\right].$$

Extensive use will be made of the following lemma. Its proof is a simple extension of the one given for n=1 in Yoshizawa [8].

LEMMA 1.1. Suppose that $y(x)$ is a solution of (1.1), and that for some $M \ge 0$ and $[\alpha,\beta] \subseteq [a,c]$, $V_M(x,y_1,\dots,y_n)$ is a control function. Then $V_M(x,y(x),\dots,y^{(n-1)}(x))$ is nondecreasing, (nonincreasing), if and only if $V_M'(x,y(x),\dots,y^{(n-1)}(x)) \ge 0$, $(V_M'(x,y(x),\dots,y^{(n-1)}(x)) \le 0)$.

In section 2, we carry out the construction for matching solutions of (k-1)-point boundary value problems with 2-point boundary value problems in obtaining a solution of (1.1), (1.2). Then in section 3, results completely analogous to those obtained in section 2 are stated for solutions of (1.1), (1.3).

2. EXISTENCE OF SOLUTIONS OF (1.1), (1.2).

In this section, for $2 < k \le n$, let $m_1,\dots,m_k$ be positive integers such that $\sum_{i=1}^{k} m_i = n$ and $m_k = 1$. Let $a \le x_1 < \dots < x_{k-1} = b < x_k \le c$ and $y_{ij} \in \mathbb{R}$,

$0 \le i \le m_j -1$, $1 \le j \le k$ be given. We match solutions $y(x)$ of (1.1) satisfying

$$y^{(i)}(x_j) = y_{ij},\ 0 \le i \le m_j -1,\ 1 \le j \le k-1,\ y^{(n-1)}(x_{k-1})=m, \tag{2.1}$$

with solutions $z(x)$ of (1.1) satisfying

$$z^{(i)}(x_{k-1}) = y^{(i)}(x_{k-1}),\ 0 \le i \le n-3,\ z^{(n-1)}(x_{k-1})=m,\ z(x_k)= y_{0,k}, \tag{2.2}$$

where $m \in R$, to obtain a solution of (1.1), (1.2). The use made of a family of control functions in establishing existence of unique solutions of (k-1)-point and 2-point problems and in establishing desired monotone properties is seen in the next four theorems.

THEOREM 2.1. Assume that there exists a control function $V_0(x,y_1,\ldots,y_n)$ on $[a,b]$ such that $V_0'(x,y(x),\ldots,y^{(n-1)}(x)) \ge 0$, for all solutions $y(x)$ of (1.1). Then for each $m \in R$, the boundary value problem for (1.1) satisfying

$$y^{(i)}(x_j) = y_{ij},\ 0 \le i \le m_j -1,\ 1 \le j \le k-1,\ y^{(n-2)}(x_{k-1})=m, \tag{2.3}$$

has a unique solution.

PROOF. It suffices to show that the boundary value problem for (1.1) satisfying $y^{(i)}(x_j) = 0$, $0 \le i \le m_{j-1}$, $1 \le j \le k-1$, $y^{(n-2)}(x_{k-1}) = 0$, has only the trivial solution. Assume on the contrary that this boundary value problem has a nontrivial solution $y(x)$. It follows that there exist points $x_1 < \tau_1 < \tau_2 < \tau_3 \le x_k$ such that $y^{(n-2)}(\tau_1)= y^{(n-2)}(\tau_3)= y^{(n-1)}(\tau_2)= 0$, and $y^{(n-2)}(x)$ or $-y^{(n-2)}(x)$ has a positive local maximum at $x = \tau_2$.

Assume without loss of generality that $y^{(n-2)}(x)$ has a positive local maximum at $x = \tau_2$. Now from our hypotheses, $V_0(\tau_1, y(\tau_1),\ldots,y^{(n-1)}(\tau_1))$ $= V_0(\tau_3, y(\tau_3),\ldots,y^{(n-1)}(\tau_3)) = 0$ and $V_0(\tau_2, y(\tau_2),\ldots,y^{(n-1)}(\tau_2)) > 0$. However, since $V_0'(x,w(x),\ldots,y^{(n-1)}(x)) \ge 0$, $V_0(x,y(x),\ldots,y^{(n-1)}(x))$ is nondecreasing; consequently, $V_0(\tau_3, y(\tau_3),\ldots,y^{(n-1)}(\tau_3)) > 0$ which is a contradiction. Thus, the assertion of the theorem is true.

THEOREM 2.2. Assume that for each $m \in R$, there exists a solution $y_1(x,m)$ of boundary value problem (1.1), (2.1). If for each $M \ge 0$, there exists a control function $V_M(x,y_1,\ldots,y_n)$ on $[a,b]$ such that $V_M'(x,y(x),\ldots, y^{(n-1)}(x)) \ge 0$ for all solutions $y(x)$ of (1.1), then $y_1^{(n-2)}(x_{k-1},m)$ is a strictly increasing function of m.

PROOF. Let $m_1 < m_2$ and assume $y_1^{(n-2)}(x_{k-1},m_2) \le y_1^{(n-2)}(x_{k-1},m_1)$. Then consider the nontrivial solution $w(x) \equiv y_1(x,m_1) - y_1(x,m_2)$ of (1.1). It follows that $w^{(n-2)}(x_{k-1}) \ge 0$.

Since $w^{(i)}(x_j) = 0$, $0 \le i \le m_j -1$, $1 \le j \le k-1$, it follows by successive applications of Rolle's Theorem that there exists $\tau_1 \varepsilon (x_1, x_{k-1})$ such that $w^{(n-2)}(\tau_1)=0$. Yet, from $w^{(n-2)}(\tau_1) = 0$, the assumption that $w^{(n-2)}(x_{k-1}) \geq 0$, and the fact that by construction $w^{(n-1)}(x_{k-1}) = m_1 - m_2 < 0$, we have that there exists $\tau_2 \varepsilon (\tau_1, x_{k-1})$ such that $w^{(n-1)}(\tau_2) = 0$ and $w^{(n-2)}(x)$ has a positive local maximum at $x = \tau_2$. Now, let $\tau_1 < \alpha < \tau_2 < \beta < x_{k-1}$ be such that $w^{(n-2)}(\alpha) = w^{(n-2)}(\beta) = M$ and $w^{(n-2)}(\tau_2) > M$. Since there exists a control function V_M, we have $V_M(\alpha, w(\alpha), \ldots, w^{(n-1)}(\alpha)) = V_M(\beta, w(\beta), \ldots, w^{(n-1)}(\beta)) = 0$ and $V_M(\tau_2, w(\tau_2), \ldots, w^{(n-1)}(\tau_2)) > 0$. However, since $V\ (x, w(x), \ldots, w^{(n-1)}(x))$ is nondecreasing on $[a,b]$, it follows that $V_M(\beta, w(\beta), \ldots, w^{(n-1)}(\beta)) > 0$; again a contradiction.

Thus, $w^{(n-2)}(x_{k-1}) < 0$, which in turn implies that $y_1^{(n-2)}(x_{k-1}, m_1) < y_1^{(n-2)}(x_{k-1}, m_2)$. The proof is complete.

REMARK. We remark that, under the hypotheses of Theorem 2.2, it can be argued by finite induction that $y_1^{(n-j)}(x_{k-1}, m)$, $2 \leq j \leq n-m_{k-1}$, are all strictly increasing functions of m.

The next two theorems follow from arguments very similar to those used in Theorems 2.1 and 2.2.

THEOREM 2.3. Let $r_{m_{k-1}}, \ldots, r_{n-3}$ be given. Assume that there exists a control function $W_0(x, y_1, \ldots, y_n)$ on $[b,c]$ such that $W_0'\ (x, y(x), \ldots, y^{(n-1)}(x)) \geq 0$ for all solutions $y(x)$ of (1.1). Then, for each $m \varepsilon R$, the boundary value problem for (1.1) satisfying

$$y^{(i)}(x_{k-1}) = \begin{Bmatrix} y_{i,k-1}, & 0 \leq i \leq m_{k-1}-1 \\ \\ r_i & , m_{k-1} \leq i \leq n-3 \end{Bmatrix} , y^{(n-2)}(x_{k-1}) = m,\ y(x_k) = y_{0,k}, \qquad (2.4)$$

has a unique solution.

THEOREM 2.4. Let $y_1(x,m)$ be as in Theorem 2.2 and assume that for each $m \varepsilon \mathbf{R}$, there exists a solution $y_2(x,m)$ of boundary value problem (1.1), (2.2). If for each $M \geq 0$, there exists a control function $W_M(x, y_1, \ldots y_n)$ on $[b,c]$ such that $W_M'\ (x, y(x), \ldots, y^{(n-1)}(x)) \geq 0$ for all solutions $y(x)$ of (1.1), then $y_2^{(n-2)}(x_{k-1}, m)$ is a strictly decreasing function of m.

PROOF. Let $m_1 < m_2$ and then set $w(x) \equiv y_2(x, m_2) - y_2\ (x, m_1)$. Then argue as in Theorem 2.2 that $w^{(n-2)}(x_{k-1}) < 0$.

We are now prepared to match solutions and obtain a solution of (1.1), (1.2).

THEOREM 2.5. Assume that, for each $m \in \mathbb{R}$, there exists a unique solution of (1.1), (2.1) on $[a,b]$, and that, for each $m \in \mathbb{R}$, there exists a solution of (1.1), (2.2) on $[b,c]$. Assume, moreover, that the boundary value problem for (1.1) on $[b,c]$ satisfying

$$y^{(i)}(x_{k-1}) = 0,\ i = 0, \ldots, n-3, n-1,\ y(x_k) = 0, \qquad (2.5)$$

has only the trivial solution. If for each $M \geq 0$, there exist control functions $V\ (x,y\ ,\ldots,y\ )$ and $W\ (x,y\ ,\ldots,y\ )$ on $[a,b]$ and $[b,c]$ respectively, such that $V_M'(x,y(x),\ldots,y^{(n-1)}(x)) \geq 0$ and $W_M'\ (x,y(x),\ldots,y^{(n-1)}(x)) \geq 0$ for all solutions $y(x)$ of (1.1), then the k-point boundary value problem (1.1), (1.2) has a solution on $[a,c]$.

PROOF. If $y_1(x,m)$ is a solution of the boundary value problem (1.1), (2.1), then by Theorem 2.2, $y_1^{(n-2)}(x_{k-1}, m)$ is a strictly increasing function of m. We contend, furthermore, that $y_1^{(n-2)}(x_{k-1}, m)$ is a continuous function of m with range all of $\mathbb{R}$. To see this, it suffices to show the latter; that is,

$$\left\{ y_1^{(n-2)}(x_{k-1},m) \mid m \in \mathbb{R} \right\} = \mathbb{R}.$$

Thus, let $r \in \mathbb{R}$. From Theorem 2.1, there is a unique solution $u(x)$ of (1.1), (2.3) satisfying

$$u^{(i)}(x_j) = y_{ij},\quad 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,$$
$$u^{(n-2)}(x_{k-1}) = r.$$

Consider now the solution $w(x) \equiv u(x) - y_1(x,u^{(n-1)}(x_{k-1}))$ of equation (1.1). $w(x)$ satisfies the boundary conditions of type (2.1),

$$w^{(i)}(x_j) = 0,\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,$$

and

$$w^{(n-1)}(x_{k-1}) = u^{(n-1)}(x_{k-1}) - u^{(n-1)}(x_{k-1}) = 0.$$

By the hypotheses of the theorem, $w(x) \equiv 0$, and hence $u(x) = y_1(x,u^{(n-1)}(x_{k-1}))$. Consequently, $y_1^{(n-2)}(x_{k-1}, u^{(n-1)}(x_{k-1})) = u^{(n-2)}(x_{k-1}) = r$, and it follows that $r \in \left\{ y_1^{(n-2)}(x_{k-1}, m) \mid m \in \mathbb{R} \right\}$.

In summation, $y_1^{(n-2)}(x_{k-1},m)$ is a strictly increasing, continuous function of m with range all of $\mathbb{R}$.

Similarly, if as in Theorem 2.4, $y_2(x,m)$ is a solution of the boundary value problem (1.1), (2.2), then it will follow that, from the existence of unique solutions of (1.1), (2.4) and (1.1), (2.5), $y_2^{(n-2)}(x_{k-1},m)$ is a strictly decreasing, continuous function of m with range all of $\mathbb{R}$. Thus, there is a unique $m_0 \in \mathbb{R}$, such that $y_1^{(n-2)}(x_{k-1}, m_0) = y_2^{(n-2)}(x_{k-1}, m_0)$. Then

$$y(x) = \begin{cases} y_1(x,m_0) & , a \le x \le b, \\ y_2(x,m_0) & , b \le x \le c, \end{cases}$$

is a solution of the boundary value problem (1.1), (1.2) on $[a,c]$.

3. EXISTENCE OF SOLUTIONS OF (1.1), (1.3).

In this section, let k, $m_1,\ldots,m_k$, $a \le x_1 \ldots < x_{k-1} = b < x_k \le c$ and $y_{ij} \in \mathbf{R}$ be as in the previous section. Let $\mu \in \{0,1,\ldots,n-2\}$ be given. We state theorems analogous to those in section 2 in which solutions $y(x)$ of (1.1) satisfying

$$y^{(i)}(x_j) = y_{ij}\ ,\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ y^{(n-1)}(x_{k-1}) = m, \tag{2.1}$$

are matched with solutions $v(x)$ of (1.1) satisfying

$$v^{(i)}(x_{k-1}) = y^{(i)}(x_{k-1}),\ 0 \le i \le n-3,\ v^{(n-1)}(x_{k-1}) = m,\ v^{(\mu)}(x_k) = y_{0,k}, \tag{3.1}$$

where $m \in \mathbf{R}$, yielding a solution of (1.1), (1.3). We will omit the proofs of these theorems. Moreover, Theorems 2.1 and 2.2 are applicable in this section.

THEOREM 3.1. Assume the hypotheses of Theorem 2.3. Then, for each $m \in \mathbf{R}$, the boundary value problem for (1.1) satisfying

$$y^{(i)}(x_{k-1}) = \begin{Bmatrix} y_{i,k-1} & , 0 \le i \le m_{k-1}-1 \\ r_i & , m_{k-1} \le i \le n-3 \end{Bmatrix},\ y^{(n-2)}(x_{k-1}) = m,\ y^{(\mu)}(x_k) = y_{0,k}, \tag{3.2}$$

has a unique solution

Theorem 3.2. Let $y_1(x,m)$ be as in Theorem 2.2 and assume that for each $m \in \mathbf{R}$, there exists a solution $v(x,m)$ of boundary value problem (1.1), (3.1).If for each $M \ge 0$, there exists a control function $W_M(x,y_1,\ldots,y_n)$ on $[b,c]$ such that $W_M'(x,y(x),\ldots,y^{(n-1)}(x)) \ge 0$ for all solutions $y(x)$ of (1.1), then $v^{(n-2)}(x_{k-2},m)$ is a strictly decreasing function of m.

THEOREM 3.3. Assume that, for each $m \in \mathbf{R}$, there exists a unique solution of (1.1), (2.1) on $[a,b]$, and that, for each $m \in \mathbf{R}$, there exists a solution of (1.1), (3.1) on $[b,c]$. Assume, moreover, that the boundary value problem for (1.1) on $[b,c]$ satisfying

$$y^{(i)}(x_{k-1}) = 0,\ i = 0,\ldots,n-3,\ n-1,\ y^{(\mu)}(x_k) = 0,$$

has only the trivial solution. If for each $M \ge 0$, there exist control functions $V_M(x,y_1,\ldots,y_n)$ on $[a,b]$ and $W_M(x,y_1,\ldots,y_n)$ on $[b,c]$, such that $V_M'(x,y(x),\ldots,y^{(n-1)}(x)) \ge 0$ and $W_M'(x,y(x),\ldots,y^{(n-1)}(x)) \ge 0$ for all solutions $y(x)$ of (1.1), then the boundary value problem (1.1), (1.3) has a solution on $[a,c]$.

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