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ON THE NATURAL DENSITY OF THE RANGE OF THE TERMINATING NINES FUNCTION

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ABSTRACT. Noting that the expression $\sum_{t\geq 1} [\frac{n}{10^t}]$ gives the number of terminating nines which occur up to n but not including n, we will denote the above expression by t(n) and call t the "terminating nines function". The natural density of the set T= {t(n): n=1,2,3, ...} will be determined.

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1. INTRODUCTION.

The number of positive integers in a set A, not exceeding x, is denoted by A(x). The natural density, d(A), of the set A is defined as

$$d(A) = \lim_{x\to\infty} \frac{A(x)}{x},$$

provided this limit exists. The determination of the natural density of a given set of positive integers is an important topic in most number theory textbooks and is the subject of much research.

For example, the set of positive integers

$$N = \{n: s(n) \text{ is a factor of } n\},$$

where s(n) denotes the digital sum of n, is the set of Niven numbers [1] and was shown to have a natural density of 0 in [2]. Here, we are interested in a part of the digital sum function.

It has been shown that

$$s(n) = n - 9 \sum_{t \geq 1} \left[\frac{n}{10^t}\right]$$

where, as usual, the square brackets denote the integral part operator. Noting that the expression

$$\sum_{t \geq 1} \left[\frac{n}{10^t}\right] \tag{1.1}$$

gives the number of terminating nines which occur up to n but not including n, we will denote (1.1) by t(n) and call t the "terminating nines function". The natural density of the set $T = \{t(n): n = 1,2,3, \ldots\}$ will be determined in what follows. Note that T does not include every positive integer since, for example, $10 \notin T$.

2. NOTATION AND TERMINOLOGY.

In what follows, we will say that the terminating nines function, t, has a "jump" of size k at an integer a if $t(a)=t(a-1) + k$. Thus, t has a jump of size k if and only if $a - 1$ ends with exactly k nines. To determine the natural density of T, we first show that

$$\lim_{n \to \infty} \frac{T(t(n))}{t(n)} = \frac{9}{10},$$

where T(t(n)) is the number of members of T not exceeding t(n). To do this, we will count how many integers are missing from set $\{t(1), t(2), \ldots, t(n)\}$. If α_n is the number of these missing integers, then it follows that

$$T(t(n)) = t(n) - \alpha_n.$$

3. THE NATURAL DENSITY OF T.

Noting that if $1 \leq a \leq n$ and t has a jump of size k at a, then this jump will produce k-1 missing integers. Moreover, each missing integer is a result of some jump at a for $1 \leq a \leq n$. Thus, each $1 \leq a \leq n$, such that 10^k divides a but 10^{k+1} does not divide a, produces k-1 missing integers. Hence, α_n is the number of terminating 0's in all integers $1 \leq a \leq n$, minus the number of integers $1 \leq a \leq n$ which end with 0. Therefore, since

$$\alpha_n = \sum_{j \geq 1} \left[\frac{n}{10^j}\right] - \left[\frac{n}{10}\right],$$

we have that

$$T(t(n)) = \left[\frac{n}{10}\right].$$

Using the above, we thus conclude that

$$\frac{T(t(n))}{t(n)} = \frac{[\frac{n}{10}]}{[\frac{n}{10}] + [\frac{n}{10^2}] + \ldots}$$

which may be written as

$$\frac{T(t(n))}{t(n)} = \frac{\frac{n}{10} + O(1)}{\frac{n}{10} + \frac{n}{10^2} + \ldots + O(\log n)},$$

since the denominator is equal $\frac{n}{10} + \frac{n}{10^2} + \ldots + O(\log n)$, and the numerator is equal to $\frac{n}{10} + O(1)$. Thus,

$$\lim_{n \to \infty} \frac{T(t(n))}{t(n)} = \lim_{n \to \infty} \frac{\frac{n}{10} + O(1)}{\frac{n}{10} + \frac{n}{10^2} + \ldots + O(\log n)}$$

$$= \frac{9}{10}.$$

Letting x be an arbitrary integer, and y be such that

$$t(y) \leq x < t(y + 1),$$

we have that $x - t(y) = O(\log x)$ since $x - t(y)$ does not exceed the number of digits in x.

Since, $T(x) = T(t(y))$, we have

$$\frac{T(x)}{x} = \frac{T(t(y))}{x} = \frac{T(t(y))}{t(y) + O(\log x)}$$

and so, by the above limit, it follows that

$$\lim_{x \to \infty} \frac{T(x)}{x} = \frac{9}{10}.$$

Stating this as a theorem we have:

THEOREM 1. Let $T = \{ t(n): n = 1,2,\ldots \}$ where t is the terminating nines function. Then $d(T) = \frac{9}{10}$.

4. GENERALIZATION TO BASE b.

Finally, it should be noted that the development given above and Theorem 1 can be generalized to any integral base b. If $t_b(n)$ denotes the number of terminating b-1's in the base b representation of the sequence of positive integers up to n, then we have the following generalization of Theorem 1:

THEOREM 1'. Let $\{t_b(n): n = 1,2, \ldots\}$. Then $d(T) = \frac{b-1}{b}$.

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