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NOTE ON CERTAIN SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS OF ORDER α

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ABSTRACT. The object of the present paper is to show a result for functions belonging to the class $R(\alpha)$ which is the subclass of close-to-convex functions in the unit disk U.

KEY WORDS AND PHRASES. Close-to-convex of order α, class $P(\alpha)$, class $R(\alpha)$, starlikeness bound.
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1. INTRODUCTION.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$. A function $f(z)$ belonging to the class A is said to be close-to-convex of order α if and only if it satisfies the condition

$$\mathrm{Re}\{f'(z)\} > \alpha \qquad (1.2)$$

for some α $(0 \leq \alpha < 1)$ and for $z \in U$. We denote by $P(\alpha)$ the subclass of A consisting of functions which are close-to-convex of order α in the unit disk U.

Further, let $R(\alpha)$ be the subclass of A consisting of all functions which satisfy the condition

$$|f'(z) - 1| < 1 - \alpha \tag{1.3}$$

for some α $(0 \leq \alpha < 1)$ and for all $z \in U$.

It is clear that $R(\alpha) \subset P(\alpha)$ for $0 \leq \alpha < 1$. Nunokawa, Fukui, Owa, Saitoh and Seking [1] have showed the starlikeness bound of functions in the class $R(\alpha)$. Also, the starlikeness bound of functions belonging to the class $P(\alpha)$ was given by Fukui, owa, Ogawa and Nunokawa [2].

2. MAIN RESULT.

In order to prove our main result, we have to recall here the following lemma due to Lewandowski, Miller and Ziotkiewicz [3].

LEMMA. Let β be real and $|\beta| < \pi/2$. Let $\phi(u,v)$ be a complex valued function

$$\phi: D \to C,\ D \subset C \times C \qquad \text{(C is the complex plane)},$$

and lrt $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u,v)$ satisfies

(i) $\phi(u,v)$ is continuous in D;

(ii) $(e^{i\beta},0) \in D$ and $\text{Re}\{\phi(e^{i\beta},0)\} > 0$;

(iii) $\text{Re}\{\phi(iu_2,v_1)\} \leq 0$ when $(iu_2,v_1) \in D$ and

$$v_1 \leq -\frac{1 - 2u_2\sin\beta + u_2^2}{2\cos\beta}.$$

Let $p(z) = e^{i\beta} + p_1z + p_2z^2 + \dots$ be regular in the unit disk U such that $(p(z),zp'(z)) \in D$ for all $z \in U$. If

$$\text{Re}\{\phi(p(z),zp'(z))\} > 0 \qquad (z \in U),$$

then $\text{Re}\{p(z)\} > 0$ $(z \in U)$.

Applying the above lemma, we derive

THEOREM. Let the function $f(z)$ defined by (1.1) be in the class $R(\alpha)$. Then

$$\text{Re}\ \{e^{i\beta}\frac{f(z)}{z}\} > 0, \tag{2.1}$$

where

$$|\beta| < \frac{\pi}{2} - \operatorname{Sin}^{-1}(1 - \alpha). \tag{2.2}$$

PROOF. It follows from $f(z) \varepsilon R(\alpha)$ that

$$\operatorname{Re}\{e^{i\beta}f(z)\} > 0 \qquad (z \varepsilon U) \tag{2.3}$$

for $|\beta| < \pi/2 - \operatorname{Sin}^{-1}(1 - \alpha)$. Defining the function $p(z)$ by

$$e^{i\beta}\frac{f(z)}{z} = p(z), \tag{2.4}$$

we can see that $p(z) = e^{i\beta} + p_1 z + p_2 z^2 + \ldots$ is regular in U. Taking the differentiations of both sides in (2.4), we have

$$e^{i\beta}f'(z) = p(z) + zp'(z). \tag{2.5}$$

It follows from (2.3) that

$$\operatorname{Re}\{e^{i\beta}f'(z)\} = \operatorname{Re}\{p(z) + zp'(z)\} > 0. \tag{2.6}$$

Setting

$$\phi(u,v) = u + v \qquad (\text{note that } u = p(z) \text{ and } v = zp'(z)), \tag{2.7}$$

we see that

(i) $\phi(u,v)$ is continuous in $D = C \times C$;

(ii) $(e^{i\beta},0) \varepsilon D$ and $\operatorname{Re}\{\phi(e^{i\beta},0\} = \cos\beta > 0$;

(iii) for all $(iu_2,v_1) \varepsilon D$ such that

$$v_1 \leq -\frac{1 - 2u_2\sin\beta + u_2^2}{2\cos\beta},$$

$$\begin{aligned}\operatorname{Re}\{\phi(iu_2,v_1)\} &= v_1 \\ &\leq -\frac{1 - 2u_2\sin\beta + u_2^2}{2\cos\beta} \\ &\leq 0.\end{aligned}$$

Therefore, the function $\phi(u,v)$ defined by (2.7) satisfies the conditions in Lemma.

Using Lemma, we have

$$\mathrm{Re}\{p(z)\} = \mathrm{Re}\,\{e^{i\beta}\frac{f(z)}{z}\} > 0$$

which completes the proof of Theorem.

Letting $\alpha = 0$ in Theorem, we have

COROLLARY. Let the function f(z) defined by (1.1) be in the class R(0). Then

$$\mathrm{Re}\{\frac{f(z)}{z}\} > 0 \qquad (z \in U).$$

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