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RESEARCH EXPOSITORY AND SURVEY ARTICLES

PRIME VALUED POLYNOMIALS AND CLASS NUMBERS OF QUADRATIC FIELDS

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ABSTRACT. It is the purpose of this paper to give a survey of the relationship between the class number one problem for real quadratic fields and prime-producing quadratic polynomials; culminating in an overview of the recent solution to the class number one problem for real quadratic fields of Richaud-Degert type. We conclude with new conjectures, questions and directions.

KEY WORDS AND PHRASES. Class number one, real quadratic fields, Richaud-Degert types, prime-valued quadratic polynomials, Gauss' conjecture.
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1. INTRODUCTION

There has long been a certain fascination with prime valued polynomials. For example in 1772 Euler [1] found that $x^2 - x + 41$ is prime for all integers x with $1 \leq x \leq 40$. Later we will see that this is not so much a property of the polynomial as it is of its discriminant -163. It happens that the more general polynomial $q(x) = x^2 - x + (p + 1)/4$ is prime for all integers x with $1 \leq x \leq (p-3)/4$ where all integers x with $1 \leq x \leq (p-3)/4$ where $p \in \{7,11,19,43,67,163\}$. The polynomial $q(x)$ is related to Gauss' class number one problem for complex quadratic fields. Herein we are concerned with describing this and other such relationships including recently discovered prime quadratics related to *real* quadratic fields. Before we do this, we briefly outline the meaning and history of the class group and class number. We shall understand a number field to be a finite extension of the rational number field Q.

Kummer's work on Fermat's last theorem led him to the observation that the rings of integers O_K of certain number fields K (actually cyclotomic ones), did not have the property of unique factorization of elements into a product of prime elements. Dedekind restored, in a sense, unique factorization by introducing the notion of an *ideal.* In an integral domain D with quotient field K, a *fractional* ideal is a D-submodule I of K for which there exists a non-zero $\alpha \in D$ such that $\alpha \cdot I \subseteq D$. In what we now call a *Dedekind domain* every non-zero fractional ideal may be uniquely written as a product of powers of distinct prime ideals. Hence the monoid, I of non-zero fractional ideals of a Dedekind domain is a group. The *principal* fractional ideals (i.e., those ideals with a single generator) form a subgroup P of I. The quotient group I/P is called the *ideal class group* of the Dedekind domain. It is a fact that rings of integers O_K of number fields K are Dedekind domains. We let C_K denote the ideal class group of O_K (or simply of K). Dirichlet proved that C_K is finite. We refer to its order h_K as the *class number* of K. Moreover we see that O_K is a principal ideal domain (P.I.D.) if and only if $h_K = 1$. It is a well-known fact that O_K is a unique factorization domain (U.F.D.) if and only if it is a P.I.D. Thus Kummer's essential obstruction in his investigation of Fermat's last theorem were cyclotomic fields with class number bigger than one. In fact Fermat's last theorem is true for a prime $p > 2$ if p does not divide the class number of the p-th cyclotomic field $Q(\zeta_p)$, (where ζ_p is a primitive p-th root of unity). Class numbers bigger than one somehow measure how far away O_K is from being a U.F.D.

QUESTION: If $h_k > h_L$ then does this mean that K is farther away (in some sense) from being a UFD then L is? In his survey article [2] Masley cites well-known examples of Furtwangler to conclude in the negative and says: "The meaning of class numbers larger than 2 is then a complete mystery". However the Furtwangler examples look at "activity' in the Hibert Class field. In fact, in response to a problem stated by Narkiewicz in 1974 (to arithmetically characterize all algebraic number fields with class number bigger than 2), David rush solved the problem in terms of elementary factorization properties in 1983. The result is too technical to state here but a result of U. Krause for the cyclic case shows the flavour of the approach as follows. (The term x *primary* should be understood to mean $x|yz$ implies $x|y$ or $x|z^n$ for some integer $n \geq 1$).

THEOREM. C_K is cyclic of prime power order if and only if there exists on $m \geq 0$ such that the m^{th} power of every irreducible integer is a product of at most m primary integers. h_K is given by the smallest possible m.

Another result (attributable to Narkiewicz) which examines the $h_K > h_L$ phenomenon is as follows.

Let $F_K(x)$ be the number of non-associated integers a of K with unique factorization and $|N_{K|Q}(a)| \leq x$. Then we have $h_K > h_L$ if and only if $F_K(x)/F_L(x) = 0$.

In Fact. If $A_K(x) = \log(F_K(x)/\log x)$ then $\lim_{x\to\infty} (A_K(x)/\log\log x) = 1 - (1/h_K)$.

CONCLUSION. The answer to the Question depends upon what you mean by "farther away from". The answer is clearly "yes" in terms of elementary factorization criteria such as that of Krauss (above) or D. Rush; or in terms of "density" as with $F_K(x)$ and $A_K(x)$ as above. The answer is not so clear if you look outside K in terms of the principal ideal theorem of class field theory as Masley interpreted the Furtwangler examples.

2. COMPLEX QUADRATIC FIELDS AND PRIME QUADRATICS

Heilbronn and Linfoot [3] proved that there are at most ten complex quadratic fields with class number one; namely $Q(\sqrt{-d})$ for $d \in \{1,2,3,7,11,19,43,67,163$, and *possibly* one other}. Baker [4] and Stark [5] independently eliminated the other potential d. For a complete survey of the solution to the class number one problem see Goldfeld [6]. Also included therein is the history of the solution to a more general problem going back to Guass (i.e., to give an effective lower bound for discriminants of all complex quadratic fields having a given class number). The 1987 Cole prize in number theory was jointly awarded to D. Goldfeld, B. Gross and D. Zagier for their solution to this problem (see [7, pp.232–234]).

Now we return to the prime quadratics introduced earlier. In 1913 Rabinovitch obtained:

THEOREM 2.1. (Rabinovitch [8] and [9]). Let $d \equiv 3 \pmod 4$, $d > 0$ and $K = Q(\sqrt{-d})$. Then $p(x) = x^2 - x + (d+1)/4$ is prime for all integers x with $1 \le x \le (d-3)/4$ if and only if $h_K = 1$.

Theorem 2.1 together with the aforementioned solution to Gauss' class number one problem for complex quadratic fields yields the remarkable property:

(P1) If $d \equiv 3 \pmod 4$ is a positive integer then $x^2 - x + (d+1)/4$ is prime for all integers x with $1 \le x \le (d-3)/4$ if and only if $d \in \{7,11,19,43,67,163\}$.

As an illustration of (P1) for $d = 163$ we get Euler's celebrated polynomial:

EXAMPLE 2.1. $x^2 - x + 41$ is prime for all integers x with $1 \le x \le 40$.

We now see the reason for the comment at the outset of the article that this is not a property of the quadratic but of its discriminant -163. For an interesting, (albeit older) note on the subject see Lehmer [10]. We now turn to the relationship between class number 2 for complex quadratic fields and certain prime quadratics. Baker [11] and Stark [12] proved that there are exactly eighteen complex quadratic fields $Q(\sqrt{-d}) = K$ with $h_K = 2$. They occur for $d \in \{5,6,10,13,15,22,35,37,51,58,91,115,123,187,235,267,403,427\}$. Prime quadratics in relation to complex quadratic fields of class number 2 were discovered by Hendy [13]:

THEOREM 2.2. Let $K = Q(\sqrt{-d})$.

(I) If $d = 2p$ where p is an odd prime then $h_K = 2$ if and only if $f(x) = 2x^2 + p$ is prime for all x with $0 \le x < \sqrt{p/2}$.

(II) If $d = p \equiv 1 \pmod 4$ is prime then $h_K = 2$ if and only if $f(x) = 2x^2 + 2x + (p+1)/2$ is prime for all integers x with $0 \le x < (\sqrt{(p-1)/2}$.

(III) If $d = pq \equiv 3 \pmod 4$, and $p < q$ primes then $h_K = 2$ if and only if $f(x) = px^2 + px + (p+q)/4$ is prime for all integers x with $0 \le x < (\sqrt{pq/12}) - \frac{1}{2}$.

We note that from the genus theory of Gauss $h_K = 2$ for $K = Q(\sqrt{-d})$ if and only if d is one of types I–III in Theorem 1.2. Hence Theorem 1.2 together with the solution to the class number two problem for complex quadratic fields yields the next remarkable relationship with prime quadratics.

(P2) If p is an odd prime then $f(x) = 2x^2 + p$ is prime for all integers x with $0 \le x < \sqrt{p/2}$ if and only if $p \in \{3,5,11,29\}$.

(P3) If $p \equiv 1 \pmod 4$ is prime then $f(x) = 2x^2 + 2x + (p + 1)/2$ is prime for all integers x with $0 \leq x < \sqrt{p-1}/2$ if and only if $p \in \{5,13,37\}$.

(P4) If $p < q$ are primes with $pq \equiv 3 \pmod 4$ then $f(x) = px^2 + px + (p + q)/4$ is prime for all integers x with $0 \leq x < (\sqrt{pq/12}) - \frac{1}{2}$ if and only if $pq \in \{15,35,51,91,115,123,187,235,267,403,427\}$.

As an illustration of the above we have:

EXAMPLE 2.2. $2x^2 + 29$ is prime for all integers $x \in$ with $0 \leq x \leq 14$.

EXAMPLE 2.3. $2x^2 + 2x + 19$ is prime for all integers $x \in$ with $0 \leq x \leq 3$.

EXAMPLE 2.4. $7x^2 + 7x + 17$ is prime for all integers $x \in$ with $0 \leq x \leq 5$.

There is only one other class of fields for which there is a complete answer to the class number two problem. There are exactly two cyclotomic fields $K = Q(\zeta_n)$ (where $n \not\equiv 2 \pmod 4$) such that $h_K = 2$. They occur for $n = 39$ and 56, (see Masley [14] for a survey of small class groups for abelian number fields.)

The solution of the class number one and two problems for complex quadratic fields led to a neat set of solutions (P1)–(P4), for related prime quadratics. The story is not so complete for real quadratic fields as we will see in the next section.

3. REAL QUADRATIC FIELDS AND PRIME QUADRATICS

The complete solution given in §2 does not yet have an analog for real quadratic fields. This is true because, at this juncture in mathematical history very little is known about class numbers of real quadratic fields. For example it is still not known whether there exist finitely many real quadratic fields with a given class number. In particular an open conjecture of Gauss says that there are infinitely many real quadratic fields with class number one. As Goldfeld said in his response to the receipt of the Cole prize (op. cit.): "This problem appears quite intractable at the moment." In point of fact we do not yet know whether there are infinitely many *number fields* with class number one. However with respect to prime quadratics some progress has been made. We begin with the introduction of a restricted class of real quadratic fields, which have been a topic of interest from several perspectives for some time.

If $d = \ell^2 + r$ where $\ell > 0$ is an integer and r divides 4ℓ with integer r such that $-\ell < r \leq \ell$, then $Q(\sqrt{d})$ (or simply d) is said to be of *Richaud–Degert type* (or (R–D)–type), (see [15] and [16]). If $|r| \in \{1,4\}$ then d is said to be of *narrow* (R–D)–type. In the general case they are called (wide) (R–D)types. In [17], S. Chowla conjectured that primes p of narrow (R–D)type $\ell^2 + 1$ with $\ell > 26$ satisfied $h(p) > 1$, where $h(p) = h_K$ for $K = Q(\sqrt{p})$. Several attempts have been made at solving this conjecture, and we now wish to link this investigation with our search for prime quadratics.

A step toward a real analog of Rabinovitch's Theorem is the following result of Kutsuna [18]:

THEOREM 3.1. If $d = 1 + 4m$ is square-free and $-x^2 + x + m$ is prime for all integers x with $1 \leq x < \sqrt{m}$ then $h(d) = 1$.

Kutsuna's result however, is incomplete in that it does not give necessary *and* sufficient conditions for $h(d) = 1$ in terms of prime quadratics. Moreover it fails for some of the most interesting fields. For example it fails when m is a square thereby eliminating Chowla's conjecture. In an attempt to link the Chowla conjecture to a search for a real analogue of Rabinovitch's result Mollin [19] discovered the following pleasant connection:

THEOREM 3.2. Let $d = 4m^2 + 1$ be square-free where m is a positive integer. Then the following are equivalent.

(I) $h(d) = 1$.

(II) p is inert in $Q(\sqrt{d})$ for all primes $p < m$; (i.e. $(d/p) = -1$ for all odd primes $p < m$, where $(\ /\ )$ is the Legendre symbol; and m is odd).

(III) $f(x) = -x^2 + x + m^2 \not\equiv 0 \pmod p$ for all positive integers x and primes p satisfying $x < p < m$.

(IV) $f(x)$ is prime for all integers x with $1 < x < m$.

Note that it is known from more general results proved in Mollin [20] that if $d = \ell^2 + 1 > 17$ is square-free and $\ell \neq 2q$ for an odd prime q then $h(d) > 1$, (see also [21]). The further reduction to $4q^2 + 1$ being a prime is known by the genus theory of Gauss. In fact the reduction to $\ell = 2q$, $q > 2$ prime is known (eg. see [17, p.48]). In [19] however all such reductions are accomplished via elementary arithmetic techniques. In any case the Chowla conjecture and the above yield the following conjectures.

The overriding assumption in the conjectures is that $p = 4q^2 + 1$ is prime and $q > 2$ is prime.

CONJECTURE 3.1. $-x^2 + x + q^2$ is prime for all integers x with $1 < x < q$ if and only if $q \leq 13$.

CONJECTURE 3.2. $(p/r) = -1$ for all primes $r < q$ if and only if $q \leq 13$.

CONJECTURE 3.3. $-x^2 + x + q^2 \not\equiv 0 \pmod r$ for all positive integers x and primes r satisfying $x < r < q$ if and only if $q \leq 13$.

Although the Chowla conjecture remains open for the above case, Mollin and Williams [22] were able to prove it under the assumption of the generalized Riemann hypothesis (GRH); i.e., the Riemann hypothesis for the zeta function of $Q(\sqrt{p})$.

Further investigations by Mollin [23] revealed the following result. In what follows $(T + U\sqrt{d})/2$ denotes the fundamental unit of $Q(\sqrt{d})$ and $N((T + U\sqrt{d})/2) = \delta$ where N denotes the norm from $Q(\sqrt{d})$ to Q. For convenience sake we let $A = (T - \delta - 1)/U^2$.

THEOREM 3.3. Let $d \equiv 1 \pmod 4$ be a positive square-free integer such that $(\sqrt{d} - 1)/2 \leq A$. Then the following are equivalent.

(1) $h(d) = 1$.

(2) p is inert in $Q(\sqrt{d})$ for all primes $p < A$.

(3) $f(x) = -x^2 + x + (d-1)/4 \not\equiv 0 \pmod p$ for all positive integers x and primes p satisfying $x < p < (\sqrt{d-1})/2$.

(4) $f(x)$ is prime for all integers x with $1 < x < (\sqrt{d-1})/2$.

Theorem 3.2 is an immediate consequence of Theorem 3.3 as is the following result on the other narrow R–D types.

COROLLARY 3.1. Let $d = m^2 \pm 4 > 5$ be square-free. Then $h(d) > 1$ unless $d = 4p + 1$ where p is prime, in which case the following are equivalent:

(i) $h(d) = 1$.

(ii) q is inert in $Q(\sqrt{d})$ for all primes $q < \begin{cases} m & \text{if } d = m^2 + 4 \\ m-2 & \text{if } d = m^2 - 4. \end{cases}$

(iii) $f(x) = -x^2 + x + p \not\equiv 0 \pmod q$ for all positive integers x and primes q satisfying $q < x < \sqrt{p}$.

(iv) $f(x)$ is prime for all integers x with $1 < x < \sqrt{p}$.

In [24] Yokoi conjectured that $h(d) > 1$ when $d = q^2 + 4$ is squarefree with $q > 17$ prime. Under the assumption of the generalized Riemann hypothesis this conjecture follows from the techniques used by Mollin and Williams in [22]. Thus we have:

CONJECTURE 3.4. If q is an odd prime then $-x^2 + x + (q^2 + 3)/4$ is prime for all integers with $1 < x < (\sqrt{q^2+3})/2$ if and only if $q \leq 17$.

In [25] Mollin and Williams were able to make substantial progress and found all real quadratic fields of narrow $R-D$ type of class number one. To state the results we will label some conditions at this juncture since we will have occassion to refer to them often. In what follows d is assumed to be a positive square-free integer, and

$$\alpha = \begin{cases} (\sqrt{d-1})/2 & \text{if } d \equiv 1 \pmod 4 \\ \sqrt{d} & \text{if } d \not\equiv 1 \pmod 4 \end{cases} \quad \text{and} \quad f_d(x) = \begin{cases} -x^2 + x + (d-1)/4 & \text{if } d \equiv 1 \pmod 4 \\ -x^2 + d & \text{if } d \not\equiv 1 \pmod 4 \end{cases}.$$

CONDITIONS:

(I) p is inert in $Q(\sqrt{d})$ for all primes $p < \alpha$.

(II) $f_d(x) \not\equiv 0 \pmod p$ for all integers x and primes p such that $0 \leq x < p < \alpha$.

(III) $f_d(x)$ is prime for all integers x with $1 < x < \alpha$

(IV) $h(d) = 1$.

THEOREM 3.4. (Mollin and Williams [25]) (I) $\Leftrightarrow$ (II) $\Rightarrow$ (III) $\Rightarrow$(IV). Moreover, if $d \equiv 1 \pmod 4$ then (III) $\Rightarrow$ (II).

What is most surprising and revealing is their next result.

THEOREM 3.5. (Mollin and Williams [25]). If (III) holds for $d > 13$ then $d \equiv 1 \pmod 4$ and d is of narrow R-D type.

In view of Theorem 3.5 we may now assert that Theorem 3.3 is essentially a statement about narrow R–D types. Theorem 3.5 was also the key for Mollin and Williams to find all real quadratic fields of narrow R–D type with class number one.

d	prime values of $f_d(x)$ for $1 < x < \alpha$
2	–
3	–
5	–
6	2
7	3
11	2,7
13	11
17	–
21	3
29	5
37	7
53	7,11
77	7,13,17
101	13,19,23
173	13,23,31,37,41
197	19,29,37,43,47
293	17,31,43,53,61,67,71
437	19,37,53,67,79,83,97,103,107
677	37,59,79,97,113,127,139,149,157,163,167.

Table 3.1. $h(d) = 1$

THEOREM 3.6. (Mollin and Williams [25]). If the G.R.H. holds then (III) $\Leftrightarrow$ (IV) If and only if d is an entry on Table 2.1

COROLLARY 3.2. Assume the G.R.H. holds. All real quadratic fields $Q(\sqrt{d})$ of narrow R–D type with $h(d) = 1$ are for $d \in \{2,3,17,21,29,37,53,77,101,173,197,293,437,677\}$. (Note that 5 is not generally considered to be an R–D type since it does not fit the fundamental unit pattern. Moreover 6,7,11 are wide R–D types and $13 = 3^2 + 4 = t^2 + r$ is not an R–D type since r must be less than t).

Thus (under GRH) Corollary 3.2 verifies conjectures of Chowla, Mollin and Yokoi (see [25]).

This left open the problem for wide R–D types. Mollin and Williams were able to settle the question in [26]. Moreover they discovered some strong connections between the class number one problem and prime producing quadratic polynomials. For example:

THEOREM 3.7. (Mollin–Williams [26]). (a) Let $d = 4\ell^2 \pm 2 > 2$. If $f_d(x) = -2x^2 + d/2$ is prime or 1 for all integers x with $0 \leq x < \sqrt{d}/2$ then $h(d) = 1$. (b) Let $d = (2\ell + 1)^2 \pm 2$ with $\ell > 0$. If $f_d(x) = -2x^2 + 2x + (d-1)/2$ is prime or 1 for all integers x with $0 < x < (\sqrt{d} + 1)/2$ then $h(d) = 1$

Tables 3.2 and 3.3 illustrate Theorem 3.7(a) and (b) respectively.

d	$f_d(x) = -2x^2 + d/2$ for $0 < x < \sqrt{d}/2$
6	3
14	7,5
38	19,17,11,1
62	31,29,23,13
398	199,197,191,181,167,149,127,101,71,37.

Table 3.2.

d	$f_d(x) = -2x^2 + 2x + (d-1)/2$ for $0 < x < (\sqrt{d} + 1)/2$
7	3
11	5,1
23	11,7
47	23,19,11
83	41,37,29,17,1
167	83,79,71,59,43,23
227	113,109,101 89,73,53,29,1

Table 3.3

Observe that in Table 3.2 the top value is $d = 398$ which yields the so-called Karst polynomial $-2x^2 + 199$. Heretofore in the literature, little or no explanation has been given for the high density of primes in this and other prime-producing polynomials It is precisely this connection with the class number one problem which is the reason, (see [26] for further details).

It was not until later work in [27] where Mollin and Williams were able to prove results similar to Theorem 3.7 for the remaining R–D types. In [26] they made two conjectures concerning these R-D types, which they were able to prove in even more generality in [27]. For example, they proved the following in [27].

THEOREM 3.8. Let $d = pq$, $p < q$ where $p \equiv 3 \equiv q \pmod 4$ are primes and $d \equiv 5 \pmod 8$ If $|px^2 + px + (p-q)/4|$ is prime or 1 for all integers x with $0 \leq x < (\sqrt{d-1})/4 - \frac{1}{2}$ then $h(d) = 1$.

Despite the seemingly more general nature of Theorem 3.8, the authors are convinced that if the hypothesis of Theorem 3.8 holds then d is of R-D type. In [27] they showed that under the assumption of a suitable Riemann hypothesis the conjecture holds. For similar related theorems and conjectures see [27].

In [26] Mollin and Williams were able to invoke the generalized Riemann hypothesis (GRH) for the zeta-function of $Q(\sqrt{d})$ to find all real quadratic fields of Richaud Degert type with $h(d) = 1$. In [28] they were able to remove the GRH and proved the following.

In what follows *extended R-D type* means those forms $d = \ell^2 + r$ with r dividing 4ℓ.

THEOREM 3.9. *With possibly only one more value remaining* all real quadratic fields of Extended R–D type $Q(\sqrt{d})$ with $h(d) = 1$ are one of the 42 values of d given in the set $\{2,3,5,6,7,11,13,14,17,21,23,29,33,37,38,47,53,62,69,77,83,101,141,167,173,197,213,227,237,293,$ $398,413,437,453,573,677,717,1077,1133,1253,1293,1757\}$.

With this virtual solution of the class number one problem for real quadratic fields of ERD type Mollin and Williams have since gone on to determine (with possibly only one more value remaining) al real quadratic fields of class number one and period $k \leq 24$ where k is the period of $w = \begin{cases} (1+\sqrt{d})/2 & \text{if } d \equiv 1 (\text{mod } 4) \\ \sqrt{d} & \text{if } d \not\equiv 1 (\text{mod } 4) \end{cases}$. Also under investigation is the class number 2 problem for real quadratic fields. We now turn to the general class number one problem for real quadratic fields.

4. THE GENERAL CLASS NUMBER ONE PROBLEM.

As noted in the introduction, the prospects for resolving the Gauss conjecture concerning an infinitude of real quadratic fields of class number one seem remote at this time.

However, Mollin and Williams in [26] we able to establish a result for *non*–R–D types:

THEOREM 4.1. If $p \equiv 1$ (mod 4) is prime and the period of the continued fraction expansion of $w = (1+\sqrt{p})/2$ is 3 then $w = \langle a, \overline{b, b, 2a-1} \rangle$, its continued fraction expansion, with $a = (b+c(b^2+1)+1)/2$ and $p = 4+4bc+(b+c(b^2+1))^2$ for some integers a,b, and c. Moreover if $p \equiv 1$ (mod 8) then $h(p) = 1$ if and only if $p = 17$. Finally, if $p \equiv 5$ (mod 8) then $h(p) = 1$ if and only if all of the following are true:

(1) $bc+1$ is prime

(2) $f_p(t) = (p-1)/4-t-t^2$ is prime whenever $t \not\equiv 2^{-1}(c-1)$ (mod $bc+1$) and $t \not\equiv -2^{-1}(c+1)$ (mod $bc+1$) where $0 \leq t \leq (b+c(b^2+1)-1)/2 = a-1$

(3) $f_p(t)/(bc+1)$ is prime or 1 whenever $t \equiv 2^{-1}(c-1)$ (mod $bc+1$) or $t \equiv -2^{-1}(c+1)$ (mod $bc+1$).

Examples of *non*–R–D types to illustrate Theorem 4.1 are $p = 317$, 461, 557 and 773.

Another result for special kinds of real quadratic fields obtained in [26] is:

THEOREM 4.2. Let $d = 49n^2+78n+31$ where $n > 0$ is even and d is square–free and let $f_d(x) = d-x^2$. Then $h(d) = 1$ if and only if the following conditions hold:

(1) $10n+9$ and $6n+5$ are primes.

(2) $f_d(x)$ is a prime for all even $x \neq 3n+2$ or $5n+4$ with $0 \leq x \leq 7n+5$.

(3) $f_d(x) = 2$ prime for all odd $x \neq n+1$ with $0 \leq x \leq 7n+5$.

As an illustration of Theorem 4.2 take $d = 383$.

It is this experience gained by Mollin and Williams in [22], [25] and [26] which gave rise to the more general results in [27] concerning the connection between the class number one problem for real quadratic fields, and certain less restrictive (than those in [22], [25] and [26]) prime producing quadratic polynomials.

We observe also that, in view of Theorem 3.4 and 3.5 we get immediately that if p is inert in $Q(\sqrt{d})$ for all primes $p < (\sqrt{d-1})/2$ then d is of narrow R–D type. Therefore to explore the general case we must relax the restriction on the number of inert primes. There is nevertheless a strong connection here as evidenced by [25, Lemma 2.1] where it was shown that conditions (I) and (II) are equivalent where α is allowed to be an arbitrary positive real number.

We conclude with some questions.

(Q1) What is the general version of Theorem 4.1? (This would yield the Rabinowitch analog for real quadratic fields.)?

(Q2) Does there exist a real quadratic field analog of the Hendy result for class number 2 and prime quadratics (as discussed in the preamble to Theorem 2.2)?

(Q3) How may we relax condition (I) of §3 so that there is a result of the type given by Theorem 3.4 for the general case?

(Q4) What is the explanation of the surprising similarity (beyond obvious norm considerations) between (P2) and Theorem 3.7(a), between (P3) and Theorem 3.7(b); and between (P4) and Theorem 2.8?

It is hoped that his article generates interest in the topics discussed and that some new headway might be made in an efffort to extend the known horizons.

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