Internat. J. Math. & Math. Sci.
VOL. 13 NO. 1 (1990) 87-92

ON COMMUTATIVITY THEOREMS FOR RINGS

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(Received February 2, 1989)

ABSTRACT. Let R be an associative ring with unity. It is proved that if R satisfies the polynomial identity $[x^n y - y^m x^n, x] = 0$ $(m > 1, n \geq 1)$, then R is commutative. Two or more related results are also obtained.

KEY WORDS AND PHRASES. Commutative rings, torsion free rings, center of a ring, commutator ideal.
1980 AMS SUBJECT CLASSIFICATION CODES. 16A20, 116A70.

1. INTRODUCTION.

Throughout this paper, R will be an associative ring, Z(R) the center of R, N the set of all nilpotent elements of R, N' the set of all zero divisors of R, and C(R) the commutator ideal of R. For any pair of elements x, y in R, we set as usual $[x,y] = xy - yx$.

Recently, generalizing some results from Bell [1] Quadri and Khan [2,3]) proved that if R is a ring satisfying the polynomial identity $[xy - y^m x^m, x] = 0$ $(m > 1, n \geq 1)$, then R is commutative. In [4], Psomopoulos has shown that an s-unital ring R in which the polynomial identity $[x^n y - y^m x, x] = 0$ $(m > 1, n > 0)$ holds, must be commutative.

In this paper, motivated by the above polynomial identities, we intend to prove results on commutativity of a ring R with unity satisfying the following property:
(i) "there exist positive integers $m > 1$ and $n \geq 1$ such that $[x^n y - y^m x^n, x] = 0$ for all x,y in R".

Our property (i) can be regarded as an amalgam of those considered by the above authors.

2. PRELIMINARIES.

In preparation for the proof of our results, we first state the following well-known results.

LEMMA 2.1. (Psomopoulos [4]). Let $x, y \in R$. If $[x,y], x = 0$, the for any positive integer k, $[x^k, y] = k\, x^{k-1}[x,y]$.

LEMMA 2.2. (Nicholson and Yaqub [5]). Let R be a ring with unity 1. Suppose that for some positive integer k, $x^k\, y = 0 = (x+1)^k y$ for all x, y in R. The $y = 0$.

LEMMA 2.3. (Bell [6]). Let f be a polynomial identity in a finite number of non-commuting indeterminates with integral coefficients. Then the following are equivalent:

(i) For any ring R satisfying $f = 0$, C(R) is a nil ideal.

(ii) For every prime p, $(GF(p))_2$ fails to satisfy $f = 0$.

LEMMA 2.3. (Tong [7]). Let R be a ring with unity 1. Let $I_o^r(x) = x^r$. If $k > 1$, let $I_k^r(x) = I_{k-1}^r(x+1) - I_{k-1}^r(x)$. Then $I_{r-1}^r(x) = \frac{1}{2}(r-1)\, r! + r!x$; $I_r^r(x) = r!$, and $I_j^r(x) = 0$ for $j > r$.

3. RESULTS.

Throughout the rest of the paper, R stands for a ring with unity 1, and satisfies the property (1). Let us first note that for any x, y in R, the property (i) can also be expressed as:

$$x^n[x,y] = [x,y^m]\, x^n. \tag{3.1}$$

Then for any positive integer t, we obtain

$$x^{tn}[x,y] = x^{(t-1)n}[x,y^m]\, x^n = x^{(t-2)n}$$

$$[x,y^{m^2}]\, x^{2n} = x^{(t-3)n}\, [x, y^{m^3}]\, x^{3n} =$$

By repeating the above process and using (3.1), we get

$$x^{tn}[x,y] = [x,y^{m^t}]\, x^{tn}. \tag{3.2}$$

We also need the following two results for the proof of our main theorem.

LEMMA 3.1. Let R be a ring with unity which satisfies the property (P). Then N Z(R).

PROOF. Let $u \in N$. Then by (3.2) for any $x \in R$ and a positive integer t, we have

$$x^{tn}[x,u] = [x, u^{m^t}]\, x^{tn}.$$

But we have u as a nilpotent element, then $u^{m^t} = 0$, for sufficiently large t. Therefore, $x^{tn}[x,u] = 0$ for all x in R. Then we have $(x+1)^{tn}\,[x,u] = 0$ for all x in R. By Lemma 2.2, this implies that $[x,u] = 0$, which forces N Z(R).

LEMMA 3.2. Let R be a ring with unity which satisfies the property (i). Then C(R) Z(R).

PROOF. Replacing x by (x + 1) in (3.1) and multiplying both sides by x^n on the right and again using (3.1), we get

$$(x+1)^n [x,y]^n = x^n [x,y] (x+1)^n, \text{ for } x,y \in R. \qquad (3.3)$$

Define

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ and } E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let $x = E_{11} + E_{12} + E_{12}$ and $y = E_{11}$. Then x and y fail to satisfy (3.3) in $(GF(p))_2$, for a prime p. So by Lemma 2.3, C(R) is a nil ideal, and hence by Lemma 3.1, C(R) Z(R). This ends the proof.

In view of Lemma 3.2, it is guaranteed that the condition of Lemma 2.1 holds for each pair of elements x, y in a ring R with unity which satisfies the property (i).

THEOREM 3.1. Let R be a ring with unity 1 satisfying property

(i) Then R is commutative.

PROOF. We are given that R is a ring with unity. So R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i ($i \in I$), each of which as a homomorphic image of R, satisfing the property (i). Thus we may assume that R is a subdirectly irreducible ring satisfying (i).

Let S be the intersection of all non-zero ideals of R_i. Then clearly $S \neq (0)$.

Now, if n = 1 in the polynomial identity (3.1), we obtain $[x,y-y^m] = (x+1)[x,y] - [x,y^m](x+1) - x[x,y] - [x,y^m]x = 0$ for all $x,y \in R$. Thus R is commutative by Herstein [8, Theorem 18].

Henceforth we assume $n > 1$. Consider the positive integer $k = p^m - p$, where p is a positive integer greater than 1. Then by (3.1) we get

$$\begin{aligned} k\, x^n[x,y] &= p^m x^n[x,y] - px^n[x,y] \\ &= p^m[x,y^m]\, x^n - p\, x^n[x,y] \\ &= [x, (py)^m]\, x^n - x^n[x, (py)] \\ &= x^n\, [x,(py)] - x^n\, [x,(py)]. \end{aligned}$$

Thus $k\, x^n[x,y) = 0$, whcih on replacing x by (x+1) yields $k\,[x,y] = 0$. Now combining Lemma 3.2 with Lemma 2.1, we have $[x^k,y] = k\, x^{k-1}[x,y] = 0$. Therefore, it follows that

$$x^k \in Z(R) \text{ for all } x \text{ in } R. \qquad (3.4)$$

Now, replacing y by y^m in (3.1) we get

$$x^n[x,y^m] = [x,(y^m)^m]\, x^n. \qquad (3.5)$$

Then by Lemma 3.2, we have

$$x^n[x,y^m] = [x,y^m]\, x^n. \qquad (3.6)$$

But

$$[x,y^m] = m\, y^{m-1}[x,y]. \qquad (3.7)$$

So using (3.6) and (3.7) we obtain

$$x^n[x,y^m] = m\, y^{m-1}[x,y^m]\, x^n,$$

and

$$[x,(y^m)^m]\, x^n = m(y^m)^{m-1}[x,y^m]\, x^n$$
$$= m\, y^{m-1}y^{(m-1)^2}\, [x,y^m]\, x^n.$$

Thus (3.5) gives

$$my^{m-1}(1-y^{(m-1)^2})\, [x,y^m]\, x^n = 0. \qquad (3.8)$$

Let us replace x by (x+1) in (3.8). Then we get $m\, y^{m-1}(1-y^{(m-1)^2})\, [x,y^m]\, (x+1)^n=0$. So by Lemma 2.2, $m\, y^{m-1}\, (1-y^{(m-1)^2})\, [x,y^m] = 0$. Therefore, by Lemma 2.3 of Quadri and Khan [3], we have

$$my^{m-1}(1-y^{k(m-1)^2})\, [x,y^m] = 0. \qquad (3.9)$$

Now, let $u \in N'$. Then by (3.4), $u^{k(m-1)^2} \in N'\ \ Z(R)$, and $S\, u^{k(m-1)^2} = (0)$. Hence using (3.9) we obtain

$$m\, u^{m-1}[x,u^m]\, (1-u^{k(m-1)^2}) = 0.$$

If $m\, u^{m-1}[x,u^m] \neq 0$, then $(1-u^{k(m-1)^2}) \in N'$ and so $S=S\, (1-u^{k(m-1)^2}) = (0)$, which gives a contradiction as $S \neq (0)$. Therefore

$$m\, u^{m-1}[x,u^m] = 0.$$

Now, from (3.1) we have

$$x^{2n}[x,u] = [x,u^{m^2}]\, x^{2n}$$
$$= m\, u^{m(m-1)}[x,u^m]\, x^{2n}$$
$$= m\, u^{m-1}u^{(m-1)^2}[x,u^m]\, x^{2n}$$
$$= m\, u^{m-1}[x,\, u^m]\, u^{(m-1)^2}x^{2n}.$$

This implies that $x^{2n}[x,u] = 0$. Hence by Lemma 2.2, we obtain $[x,u] = 0$, that is $u \in Z(R)$. Therefore, $N'\ \ Z(R)$.

Clearly, for any $x \in R$, x^k and $x^{km} \in Z(R)$. Then by (3.1) for any $y \in R$, we have the identity

$$(x^k-x^{km})x^n\, [x,y] = x^k(x^n[x,y]) - x^{km}(x^n[x,y])$$
$$= x^n\, (x^k[x,y]) - (x^{km}[x,y])\, x^n$$
$$= x^n[x,x^ky] - [x,(x^ky)^m]\, x^n$$
$$= x^n[x,x^ky] - x^n[x,x^ky].$$

Therefore,

$$(x^k - x^{km})\, x^n[x,y] = 0, \text{ and}$$
$$(x-x^t)\, x^s[x,y] = 0, \qquad (3.10)$$

where $t = km-k+1$, and $s=n+k-1$.

Now, if $x^s[x,y] = 0$, then Lemma 2.2 yields $[x,y] = 0$. But $x^s[x,y] \neq 0$ gives $x-x^t \in N' \quad Z(R)$.

Therefore, $[x - x^t, y] = 0$, for all x, y in R, which implies that $[x,y] = 0$, by Theorem 18 of [9]. Thus in evey case $[x,y] = 0$.
This proves that R is commutative.

THEOREM 3.2. Let R be an s-unital ring satisfying the property (i). Then R is commutative.

PROOF. This follows from Proposition 1 of Hirano, Kobayashi and Tominaga [9] since R with unity satisfying (i) is commutative by Theorem 3.1. Finally, we present a short and easy proof of Theorem 3.1, but under an extra condition on the commutators in the ring R. We use an iteration technique as given in Tong [7].

THEOREM 3.3. Let R be a ring with unity satisfying the property (i). If every commutator in R is m! - torsion free, then R must be commutative.

PROOF. The ring R satisfies the identity

$$x^n[x,y] = [x,y^m]\, x^n, \; n > 1, \; m > 1).$$

We shall apply the iteration on y^m. As in [7], let $I_j(y) = I_j^m(y)$ for $j = 0,1,2,\ldots$.
Then the above identity can be rewritten as

$$x^n[x,y] = [x, I_o(y)]\, x^n. \tag{3.11}$$

Replacing y by (y+1) in (3.11), we obtain

$$x^n[x,y] = [x, I_o(y+1)]\, x^n.$$

Now, using Lemma 2.4, we get

$$x^n[x,y] = [x, I_o(y) + I_1(y)]\, x^n. \tag{3.12}$$

Equations (3.11) and (3.12) when combined, give

$$0 = [x, I_1(y)]\, x^n. \tag{3.13}$$

Again, let y=y+1 in (3.13). Then using Lemma 2.4 we have

$$0 = [x, I_2(y)]\, x^n.$$

Repeating the above process (m-1) times, we reach the identity

$$0 = [x, I_{m-1}(y)]\, x^n. \tag{3.14}$$

With an application of Lemma 2.4, we end up with

$$m![x,y]\, x^n = 0.$$

Now replacing x by (x+1) in the above identity, and making ue of Lemma 2.2, we have

$$m![x,y] = 0, \text{ for all } x,y \in R.$$

By every commutator in R is m!-torsion free, so $[x,y] = 0$ for all x and y in R. Therefore, R is commutative. This completes the proof.

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