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HEARING THE SHAPE OF MEMBRANES: FURTHER RESULTS

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ABSTRACT. The spectral function $\theta(t) = \sum_{m=1}^{\infty} \exp(-t\lambda_m)$, $t > 0$ where $\{\lambda_m\}_{m=1}^{\infty}$ are the eigenvalues of the Laplacian in R^n, $n = 2$ or 3, is studied for a variety of domains. Particular attention is given to circular and spherical domains with the impedance boundary conditions $\frac{\partial u}{\partial r} + \gamma_j u = 0$ on Γ_j (or S_j), $j = 1,\ldots,J$ where Γ_j and S_j, $j = 1,\ldots,J$ are parts of the boundaries of these domains respectively, while γ_j, $j = 1,\ldots,J$ are positive constants.

1. INTRODUCTION.

The underlying problems are to deduce the precise shape of membranes from the complete knowledge of the eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_m \leq \ldots \to \infty \quad \text{as } m \to \infty, \tag{1.1}$$

for the Laplace operator Δ_n in R^n, $n = 2$ or 3.

(P1): Let $\Omega = \{(r,\theta): 0 < r < a,\ 0 \leq \theta \leq 2\pi\}$ be a circular domain of radius a and boundary Γ. Suppose that the eigenvalues (1.1) are given for the eigenvalue equation $(\Delta_2 + \lambda)\, u = 0$ in Ω together with the impedance boundary conditions:

$$\left(\frac{\partial}{\partial r} + \gamma_j\right)u = 0 \qquad \text{on } \Gamma_j,\ j = 1,\ldots,J, \tag{1.2}$$

where γ_j, $j = 1,\ldots,J$ are positive constants and the boundary Γ consists of parts Γ_j, $j = 1,\ldots,J$ such that

$$\Gamma_j = \{(r,\theta): r = a,\ \alpha_j \leq \theta \leq \alpha_{j+1},\ j = 1,\ldots,J, \alpha_1 = 0, \alpha_{J+1} = 2\pi\}.$$

(P2): Let $\Omega = \{(r,\theta,\phi): 0 < r < a,\ 0 \leq \theta \leq \pi,\ 0 \leq \phi \leq 2\pi\}$ be a spherical domain of radius a and surface S. Suppose that the eigenvalues (1.1) are given for the eigenvalue equation $(\Delta_3+\lambda)u = 0$ in Ω together with the impedance boundary conditons:

$$\left(\frac{\partial}{\partial r} + \gamma_j\right) u = 0 \qquad \text{on } S_j,\ j = 1,\ldots,J \tag{1.3}$$

where the surface S consists of parts S_j, $j = 1,\ldots,J$ such that

$S_j = \{(r,\theta,\phi): r = a,\ 0 < \theta < \pi,\ \ \alpha_j < \phi < \alpha_{j+1}, j = 1,\dots,J; \alpha_1 = 0, \alpha_{J+1} = 2\pi\}.$

The object of this paper is to determine the geometry of the domains in (P1) and (P2) as well as the impedances γ_j, $j = 1,\dots,J$ from the asymptotic expansion of the spectral function

$$\theta(t) = \sum_{m=1}^{\infty} \exp(-t\lambda_m), \tag{1.4}$$

for small positive t.

Zayed [1] has recently investigated probems (P1) and (P2) in the special case when $J = 2$, that is, when the boundary Γ consists of two parts Γ_1, Γ_2 and when the surface S consists of two parts S_1, S_2. Finally, we close this introduction with the remark that the author [2,3] has recently generalized the results of [1] to the case when $\Omega \subseteq R^n$, $n = 2$ or 3 is a simply connected bounded domain with a smooth boundary.

2. CONSTRUCTION OF $\theta(t)$ FOR PROBLEM (P1).

Following the method of Kac [4] and following closely the procedure of section 2 in Zayed [1], it is easy to show that the spectral function (1.4) associated with problem (P1) is given by:

$$\theta(t) = \iint_{\Omega} G(\underset{\sim}{x}, \underset{\sim}{x};t)d\underset{\sim}{x}, \tag{2.1}$$

where $G(\underset{\sim}{x},\underset{\sim}{x}';t)$ is the Green's function for the heat equation

$$(\Delta_2 - \frac{\partial}{\partial t})\, u = 0, \tag{2.2}$$

subject to the impedance boundary conditions (1.2) and the initial condition

$$\lim_{t\to 0} G(\underset{\sim}{x},\underset{\sim}{x}';t) = \delta(\underset{\sim}{x} - \underset{\sim}{x}'), \tag{2.3}$$

where $\delta(\underset{\sim}{x} - \underset{\sim}{x}')$ is the Dirac delta function located at the source point $\underset{\sim}{x} = \underset{\sim}{x}'$. Let us write

$$G(\underset{\sim}{x},\underset{\sim}{x}';t) = G_0(\underset{\sim}{x};\underset{\sim}{x}';t) + \chi(\underset{\sim}{x},\underset{\sim}{x}';t), \tag{2.4}$$

where

$$G_0(\underset{\sim}{x},\underset{\sim}{x}';t) = (4\pi t)^{-1} \exp\{-\frac{|\underset{\sim}{x} - \underset{\sim}{x}'|^2}{4t}\}, \tag{2.5}$$

is the "fundamental solution" of the heat equation (2.2), while $\chi(\underset{\sim}{x},\underset{\sim}{x}';t)$ is the "regular solution" chosen so that $G(\underset{\sim}{x},\underset{\sim}{x}';t)$ satisfies the impedance boundary conditions (1.2).

On setting $\underset{\sim}{x} = \underset{\sim}{x}'$ we find that

$$\theta(t) = \frac{\text{area } \Omega}{4\pi t} + K(t), \tag{2.6}$$

where

$$K(t) = \iint_{\Omega} \chi(\underset{\sim}{x},\underset{\sim}{x}\ ;t)dx. \tag{2.7}$$

The problem now is to determine the asymptotic expansion of K(t) for small positive t. In what follows we shall use Laplace transform with respect to "t" and use "s^2" as the Laplace transform parameter; thus

$$\overline{G}(\underset{\sim}{x},\underset{\sim}{x}' s^2) = \int_0^{+\infty} e^{-s^2 t}\, G(\underset{\sim}{x},\underset{\sim}{x}';t)dt. \tag{2.8}$$

An application of the Laplace transform to the heat equation (2.2) shows that $\overline{G}(\underset{\sim}{x},\underset{\sim}{x}';s^2)$ satisfies the two-dimensional membrane equation

$$(\Delta_2 - s^2)\, \overline{G}(\underset{\sim}{x},\underset{\sim}{x}';s^2) = -\, \delta(\underset{\sim}{x} - \underset{\sim}{x}') \quad \text{in } \Omega, \tag{2.9}$$

together with the impedance boundary conditions (1.2). The asymptotic expansion of K(t) as $t \to 0$, may then be deduced directly from the asymptotic expansion of $\overline{K}(s^2)$ for $s \to \infty$, where

$$\overline{K}(s^2) = \iint_\Omega \overline{\kappa}(\underset{\sim}{x},\underset{\sim}{x}\ ;\ s^2)d\underset{\sim}{x}. \tag{2.10}$$

With reference to section 3 in Stewartson and Waechter [5], it can readily be shown after some reduction that the impedance boundary conditions (1.2) give

$$\overline{K}(s^2) = -\frac{a^2}{4\pi}\sum_{m=-\infty}^{\infty}\{\sum_{j=1}^{J}(\alpha_{j+1} - \alpha_j)\, f_j\ (m;s)\}, \tag{2.11}$$

where

$$f_j(m;s) = (1 + \frac{m^2}{s^2a^2})\ \{I_m(sa)K_m(sa) - \frac{I_m(sa)}{a[sI_m'(sa) + \gamma_j\ I_m(sa)]}\}$$

$$- I_m'(sa)K_m'(sa) - \frac{\gamma_j\ I_m'(sa)}{sa[s\ I_m'(sa) + \gamma_j\ I_m(sa)]}, \tag{2.12}$$

in which I_m and K_m are modified Bessel functions. The series (2.11) is slowly convergent for large positive s and it is therefore, expedient to apply a Watson transformation [5] to obtain

$$\overline{K}(s^2) \sim -\frac{a^2}{2\pi}\sum_{j=1}^{J}(\alpha_{j+1} - \alpha_j)\int_0^{+\infty} f_j(\nu;s)d\nu \quad \text{as } s \to \infty. \tag{2.13}$$

It now follows that the functions $f_j(\nu;s)$, $j = 1,\ldots,J$ may be expressed in terms of the asymptotic expansions of the modified Bessel functions and their derivatives due to Olver [6]. These expansions for $s \to \infty$ are uniformly valid in ν for $|\arg\ \nu| < \frac{\pi}{2}$.

Now, the following cases can be considered:

CASE 1. $(0 < \Gamma_j \ll 1,\ j = 1,\ldots,J)$

In this case, it can be shown for $s \to \infty$ that

$$f_j(\nu;s) \sim \frac{(\nu^2 + s^2a^2)^{1/2}}{s^2a^2}\sum_{n=0}^{\infty}\frac{A_{j,n}(\tau)}{\nu^n}, \tag{2.14}$$

where $\tau = \frac{\nu}{(\nu^2 + s^2a^2)^{1/2}}$. For n = 0,1,2,3 we deduce that

$$A_{j,0} = 0,\ A_{j,1} = -\frac{1}{2}(\tau-\tau^3),\ A_{j,2} = \tau^2(a\gamma_j - \frac{1}{2}) - \tau^4(a\gamma_j - \frac{3}{2}) - \tau^6,$$

and

$$A_{j,3} = -\tau^3(\frac{3}{8} - a\gamma_j + a^2\gamma_j^2) - \tau^5(-\frac{23}{8} + 3a\gamma_j - a^2\gamma_j^2) - \tau^7(\frac{41}{8} - 2a\gamma_j) + \frac{21}{8}\tau^9. \quad (2.15)$$

On inserting (2.14) into (2.13) we deduce after some simplification that

$$\bar{K}(s^2) = \frac{\text{length } \Gamma}{8s} + \{1 - \frac{3a}{\pi}\sum_{j=1}^{J}(\alpha_{j+1}-\alpha_j)\gamma_j\}\frac{1}{6s^2} + O(\frac{1}{s^3}) \quad \text{as } s \to \infty. \quad (2.16)$$

On inverting Laplace transforms and using (2.6) we have the formula:

$$\theta(t) = \frac{\text{area } \Omega}{4\pi t} + \frac{\text{length } \Gamma}{8(\pi t)^{1/2}} + \{1 - \frac{3a}{\pi}\sum_{j=1}^{J}(\alpha_{j+1}-\alpha_j)\gamma_j\}\frac{1}{6} + O(t^{1/2}) \text{ as } t \to 0. \quad (2.17)$$

CASE 2. $(0 < \gamma_j << 1,\ j = 1,\dots,k$ and $\gamma_j >> 1,\ j = k+1,\dots,J)$

In this case $f_j(\nu;s)$, $j = 1,\dots,k$ have the same forms (2.14) and (2.15) while $f_j(\nu;s)$, $j = k+1,\dots,J$ have the form (2.14) where

$$A_{j,0} = 0,\ A_{j,1} = \frac{\tau}{2} + \tau^3(\frac{1}{a\gamma_j} - \frac{1}{2}) - \frac{\tau^5}{a\gamma_j},$$

$$A_{j,2} = -\frac{\tau^2}{8a\gamma_j} + \tau^4(\frac{19}{8a\gamma_j} - \frac{1}{2}) - \tau^6(\frac{43}{8a\gamma_j} - \frac{1}{2}) + \frac{25}{8a\gamma_j}\tau^8,$$

and

$$A_{j,3} = -\tau^3(\frac{1}{4a\gamma_j} - \frac{1}{8}) + \tau^5(\frac{27}{4a\gamma_j} - \frac{13}{8}) - \tau^7(\frac{107}{4a\gamma_j} - \frac{27}{8})$$

$$+ \tau^9(\frac{141}{4a\gamma_j} - \frac{15}{8}) - \frac{15}{a\gamma_j}\tau^{11}. \quad (2.18)$$

Consequently, we deduce after some reduction that

$$\theta(t) = \frac{\text{area } \Omega}{4\pi t} + \frac{1}{8(\pi t)^{1/2}}\{a\sum_{j=1}^{k}(\alpha_{j+1}-\alpha_j) - \sum_{j=k+1}^{J}(\alpha_{j+1}-\alpha_j)(a+\gamma_j^{-1})\}$$

$$+ \{1 - \frac{3a}{\pi}\sum_{j=1}^{k}(\alpha_{j+1}-\alpha_j)\gamma_j\}\frac{1}{6} + O(t^{1/2}) \quad \text{as } t \to 0. \quad (2.19)$$

CASE 3. $(\gamma_j >> 1,\ j = 1,\dots,k$ and $0 < \gamma_j << 1,\ j = k+1,\dots,J)$

This case can be deduced from the previous one and yields:

$$\theta(t) = \frac{\text{area } \Omega}{4\pi t} + \frac{1}{8(\pi t)^{1/2}}\{a\sum_{j=k+1}^{J}(\alpha_{j+1}-\alpha_j) - \sum_{j=1}^{k}(\alpha_{j+1}-\alpha_j)(a+\gamma_j^{-1})\}$$

$$+ \{1 - \frac{3a}{\pi}\sum_{j=k+1}^{J}(\alpha_{j+1} - \alpha_j)\gamma_j\}\frac{1}{6} + O(t^{1/2}) \quad \text{as } t \to 0. \quad (2.20)$$

CASE 4. $(\gamma_j >> 1,\ j = 1,\dots,J)$

In this case $f_j(\nu;s)$, $j = 1,\dots,J$ have the same forms (2.14) and (2.18). Consequently we have the formula:

$$\theta(t) = \frac{\text{area } \Omega}{4\pi t} - \frac{1}{8(\pi t)^{1/2}}\{\sum_{j=1}^{J}(\alpha_{j+1}-\alpha_j)(a+\gamma_j^{-1})\} + \frac{1}{6} + O(t^{1/2}) \quad \text{as } t \to 0. \quad (2.21)$$

With reference to section 1 in Zayed [1] and the articles by Kac [4], Gottlieb

[7], Pleijel [8], and Sleeman and Zayed [9], the asymptotic expansions (2.17), (2.19), (2.20) and (2.21) may be interpreted as:

(i) Ω is a circular domain of radius a and we have the impedance boundary conditions (1.2) with small/large impedances γ_j, $j = 1,\dots,J$ as indicated in the specifications of the four respective cases, or (ii) for the first three terms, Ω is a bounded domain in R^2 of area πa^2. Let $h < \infty$ be the number of smooth convex holes in Ω.

In case 1, it has $n = \frac{3a}{\pi}\sum_{j=1}^{J}(\alpha_{j+1} - \alpha_j)\gamma_j$ holes and a boundary length of $2\pi a$ together with Neumann boundary conditions, provided h is an integer.

In case 2, it has $h = \frac{3a}{\pi}\sum_{j=1}^{k}(\alpha_{j+1} - \alpha_j)\gamma_j$ holes, the parts Γ_j, $j = 1,\dots,k$ of the boundary Γ have lengths $a\sum_{j=1}^{k}(\alpha_{j+1} - \alpha_j)$ together with Neumann boundary conditions while the other parts Γ_j, $j = k+1,\dots,J$ have lengths $\sum_{j=k+1}^{J}(\alpha_{j+1} - \alpha_j)(a+\gamma_j^{-1})$ together with Dirichlet boundary conditions.

In case 4, it has no holes (h = 0) and a boundary length of $\sum_{j=1}^{J}(\alpha_{j+1} - \alpha_j)(a + \gamma_j^{-1})$ together with Dirichlet boundary conditions.

We close this section with the remark that when J = 2 the results (2.17), (2.19), (2.20) and (2.21) are in agreement with the results of [1].

3. CONSTRUCTION OF $\theta(t)$ FOR PROBLEM (P2).

In analogy with the two dimensional membrane problem, it is clear that $\theta(t)$ associated with problem (P2) is given by:

$$\theta(t) = \iiint_\Omega G(\underset{\sim}{x},\underset{\sim}{x};t)d\underset{\sim}{x}, \tag{3.1}$$

where $G(\underset{\sim}{x},\underset{\sim}{x}';t)$ is the Green's function for the heat equation

$$(\Delta_3 - \frac{\partial}{\partial t})\, u = 0, \tag{3.2}$$

subject to the impedance boundary conditions (1.3) and the initial condition of the form (2.3). As we have done in section 2, we can write $G(\underset{\sim}{x},\underset{\sim}{x}';t)$ for problem (P2) in a form similar to (2.4), where

$$G_0(\underset{\sim}{x},\underset{\sim}{x}';t) = (4\pi t)^{-3/2}\exp\{-\frac{|\underset{\sim}{x} - \underset{\sim}{x}'|^2}{4t}\}. \tag{3.3}$$

From (2.4), (3.1) and (3.3) we find that

$$\theta(t) = \frac{\text{volume } \Omega}{(4\pi t)^{3/2}} + K(t) \tag{3.4}$$

where

$$K(t) = \iiint_\Omega \chi(\underset{\sim}{x},\underset{\sim}{x};t)d\underset{\sim}{x}. \tag{3.5}$$

An application of the Laplace transform to the heat equation (3.2) shows that $\overline{G}(\underset{\sim}{x},\underset{\sim}{x}';t)$ satisfies the three-dimensional membrane equation

$$(\Delta_3 - s^2)\overline{G}(\underset{\sim}{x},\underset{\sim}{x}';s^2) = -\,\delta(\underset{\sim}{x} - \underset{\sim}{x}') \quad \text{in } \Omega, \tag{3.6}$$

together with the impedance boundary conditions (1.3), where

$$\overline{K}(s^2) = \iiint_\Omega \overline{\kappa}(\underset{\sim}{x},\underset{\sim}{x};s^2)d\underset{\sim}{x}. \tag{3.7}$$

With reference to section 2 in Waechter [10], it can readily be shown after some reduction that the impedance boundary conditions (1.3) give

$$\overline{K}(s^2) = -\frac{a^2}{2\pi}\sum_{m=0}^{\infty}(m+\tfrac{1}{2})\{\sum_{j=1}^{J}(\alpha_{j+1}-\alpha_j)f_j(m;s)\}, \tag{3.8}$$

where $f_j(m;s)$ have the same form (2.12) with m replaced by $m+\frac{1}{2}$.

The series (3.8) if fact diverges since $K(t) \sim \frac{1}{t}$ for small positive t; however, this difficulty may be easily removed by considering the asymptotic expansion for large positive s of

$$\overline{K}_N(s^2) = -\frac{a^2}{2\pi}\sum_{m=0}^{N}(m+\tfrac{1}{2})\{\sum_{j=1}^{J}(\alpha_{j+1}-\alpha_j)f_j(m;s)\}. \tag{3.9}$$

Inversion of the Laplace transform gives $K_N(t)$ and we may then write

$$K(t) = \lim_{N\to\infty} K_N(t). \tag{3.10}$$

On applying a Watson transformation [10] to (3.9), we find that

$$\overline{K}_N(s^2) \sim -\frac{a^2}{2\pi}\sum_{j=1}^{J}(\alpha_{j+1}-\alpha_j)\int_0^N \nu f_j(\nu;s)d\nu \quad \text{as } s\to\infty. \tag{3.11}$$

Now, the four respective cases considered in section 2, can be applied as follows:

CASE 1. $(0 < \gamma_j \ll 1,\ j = 1,\dots,J)$

On inserting (2.14) and (2.15) into (3.11) and integrating and letting $N\to\infty$, we deduce after some simplification that

$$K(t) = \frac{\text{surface area } S}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}}\{2a^2\sum_{j=1}^{J}(\alpha_{j+1}-\alpha_j)(\tfrac{1}{a}-3\gamma_j)\} + O(t^{1/2}) \quad \text{as } t\to 0. \tag{3.12}$$

From (3.4) and (3.12) we have the formula

$$\theta(t) = \frac{\text{volume } \Omega}{(4\pi t)^{3/2}} + \frac{\text{surface area } S}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}}\{2a^2\sum_{j=1}^{J}(\alpha_{j+1}-\alpha_j)(\tfrac{1}{a}-3\gamma_j)\} + O(t^{1/2}) \quad \text{as } t\to 0. \tag{3.13}$$

CASE 2. $(0 < \gamma_j \ll 1,\ j = 1,\dots,k$ and $\gamma_j \gg 1,\ j = k+1,\dots,J)$

On inserting (2.14), (2.15) and (2.18) into (3.11) and integrating and letting $N\to\infty$ we have the formula

$$\theta(t) = \frac{\text{volume } \Omega}{(4\pi t)^{3/2}} + \frac{1}{16\pi t}\{2a^2\sum_{j=1}^{k}(\alpha_{j+1}-\alpha_j) - 2a\sum_{j=k+1}^{J}(\alpha_{j+1}-\alpha_j)(a-2\gamma_j^{-1})\}$$

$$+ \frac{1}{12\pi^{3/2}t^{1/2}} \{2a^2 \sum_{j=1}^{k} (\alpha_{j+1} - \alpha_j)(\frac{1}{a} - 3\gamma_j) + 2a \sum_{j=k+1}^{J} (\alpha_{j+1} - \alpha_j)\}$$

$$+ O(t^{1/2}) \quad \text{as } t \to 0. \tag{3.14}$$

CASE 3. ($\gamma_j >> 1$, $j = 1,\ldots,k$ and $0 < \gamma_j << 1$, $j = k+1,\ldots,J$)

This case can be deduced from the previous one and yields

$$\theta(t) = \frac{\text{volume } \Omega}{(4\pi t)^{3/2}} + \frac{1}{16\pi t} \{2a^2 \sum_{j=k+1}^{J} (\alpha_{j+1} - \alpha_j) - 2a \sum_{j=1}^{k} (\alpha_{j+1} - \alpha_j)(a - 2\gamma_j^{-1})\}$$

$$+ \frac{1}{12\pi^{3/2}t^{1/2}} \{2a^2 \sum_{j=k+1}^{J} (\alpha_{j+1} - \alpha_j)(\frac{1}{a} - 3\gamma_j) + 2a \sum_{j=1}^{k} (\alpha_{j+1} - \alpha_j)\}$$

$$+ O(t^{1/2}) \quad \text{as } t \to 0. \tag{3.15}$$

CASE 4. ($\gamma_j >> 1$, $j = 1,\ldots,J$)

On inserting (2.14) and (2.18) into (3.11) and integrating and letting $N \to \infty$ we have the formula

$$\theta(t) = \frac{\text{volume } \Omega}{(4\pi t)^{3/2}} - \frac{1}{16\pi t} \{2a \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j)(a - 2\gamma_j^{-1})\} + \frac{a}{3(\pi t)^{1/2}} + O(t^{1/2})$$

$$\text{as } t \to 0. \tag{3.16}$$

With reference to section 1 in [1] and the articles by Gottlieb [7], Waechter [10], Pleijel [11], and Zayed [12] the asymptotic expansions (3.13) - (3.16) may be interpreted as (i) Ω is a spherical domain of radius a and we have the impedance boundary conditions (1.3) with small/large impedances γ_j, $j = 1,\ldots,J$ as indicated in the specifications of the four respective cases, or (ii) for the first three terms, Ω is a bounded domain in R^3 of volume $\frac{4}{3}\pi a^3$.

In case 1, it has a surface S of area $4\pi a^2$, the parts S_j, $j = 1,\ldots,J$ of the surface S have areas $2a^2 \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j)$ and mean curvatures $(\frac{1}{a} - 3\gamma_j)$, $j=1,\ldots,J$ together with Neumann boundary conditions.

In case 2, the parts S_j, $j = 1,\ldots,k$ of the surface S have areas $2a^2 \sum_{j=1}^{k} (\alpha_{j+1} - \alpha_j)$ and mean curvatures $(\frac{1}{a} - 3\gamma_j)$ $j = 1,\ldots,k$ together with Neumann boundary conditions, while the other parts S_j, $j = k+1,\ldots,J$ have areas $2a \sum_{j=k+1}^{J} (\alpha_{j+1} - \alpha_j)(a - 2\gamma_j^{-1})$ and mean curvature $\frac{1}{a}$ together with Dirichlet boundary conditions.

In case 4, it has a surface of area $2a \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j)(a - 2\gamma_j^{-1})$ and mean curvature $\frac{1}{a}$ together with Dirichlet boundary conditions.

Finally, we note that when J = 2 the results (3.13) - (3.16) are in agreement with the results of [1].

4. DISCUSSIONS.

This paper represents a sensible extension of the author's previous publication [1] when the boundary Γ in R^2 or the surface S in R^3 consists of two parts (J = 2) to the case when Γ or S consists of J parts, where J is a finite positive integer, in which a great deal of technical analysis has gone into obtaining the results. Zayed [2,3] has recently generalized the results of [1] to the case when $\Omega \subseteq R^n$, n = 2 or 3 is a simply connected bounded domain, where a considerable amount of mathematical work has gone into obtaining the results. With reference to the previous work (See [2], [3], [11], [12]), we conclude that, there are technical difficulties and a considerable amount of mathematical work in extending the results of the present paper to the type of domains considered in [2] and [3]. This extension is still an open problem which will be discussed in a forthcoming paper.

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