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A NOTE ON NEIGHBORHOODS OF ANALYTIC FUNCTIONS HAVING POSITIVE REAL PART

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ABSTRACT. Let P denote the set of all functions analytic in the unit disk $D = \{z \mid |z| < 1\}$ having the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ with $\mathrm{Re}\{p(z)\} > 0$. For $\delta \geq 0$, let $N_\delta(p)$ be those functions $q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k$ analytic in D with $\sum_{k=1}^{\infty} |p_k - q_k| \leq \delta$. We denote by P' the class of functions analytic in D having the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ with $\mathrm{Re}\{[zp(z)]'\} > 0$. We show that P' is a subclass of P and determine δ so that $N_\delta(p) \subset P$ for $p \in P'$.

KEY WORDS AND PHRASES. Functions having positive real part (Carathéodory class), subordinate function, δ-neighborhood, and convolution (Hadamard product).
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1. INTRODUCTION

Let H denote the class of functions f analytic in the unit disk $D = \{z \mid |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$. For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in H and $\delta \geq 0$, let the δ-neighborhood of f be given by $N_\delta(f) = \{g(z) = z + \sum_{k=2}^{\infty} b_k z^k \mid \sum_{k=2}^{\infty} k|a_k - b_k| \leq \delta\}$. For $h(z) = z$, Goodman [1] has shown that $N_1(h) \subset S^*$ where S^* denotes the class of univalent functions in H which are starlike with respect to the origin. St. Ruscheweyh [2] proved that if $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ lies in C, where C denotes the class of convex univalent functions in H, then $N_\delta(f) \subset S^*$ for $\delta_n = 2^{-2/n}$. Fournier [3] found that if C were replaced by

$$\tilde{T} = \{g \in C \mid |\frac{zg''(z)}{g'(z)}| < 1, z \in D\}$$

and S^* by

$$T = \{g \in S^* \mid |\frac{zg'(z)}{g(z)} - 1| < 1, z \in D\}$$

then $N_{\delta_n}(f) \subset T$ for $\delta_n = e^{-1/n}$. Brown [4] extended the results of St. Ruscheweyh and Fournier and provided simpler proofs. We shall focus on a class of functions directly related to S^* and to other classes of univalent functions. Let P denote the class of functions analytic in $|z| < 1$ having the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ with $\mathrm{Re}\{p(z)\} > 0$ for $|z| < 1$. This family is usually called the Carathéodory class. For f in H, recall that $f \in S^*$ if and only if $p(z) = zf'(z)/f(z)$ lies in P.

Let P' denote the class of functions analytic in $|z| < 1$ having the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ with $\mathrm{Re}\{[zp(z)]'\} > 0$ for $|z| < 1$. In this paper we shall define a neighborhood of $p \in P'$ and determine $\delta > 0$ so that $N_\delta(p) \subset P$.

2. PRELIMINARY RESULTS.

We begin by defining P and P' in terms of subordination. Recall that g is subordinate to h, written $g \prec h$, if $g(z) = h(w(z))$ where w is analytic in $|z| < 1$, $w(0) = 0$ and $|w(z)| < 1$ for $|z| < 1$. Since $\frac{1+z}{1-z}$ has positive real part in $|z| < 1$, is univalent, and is 1 when $z = 0$, it is not difficult to show that

$$p \in P \text{ if and only if } p(z) \prec \frac{1+z}{1-z} \tag{2.1}$$

and that

$$p \in P' \text{ if and only if } [zp(z)]' \prec \frac{1+z}{1-z}. \tag{2.2}$$

One can also show that $P' \subset P$. For according to (2.2), if $p \in P'$ then

$$[zp(z)]' \prec \frac{1+z}{1-z}$$

and thus we have

$$\frac{[zp(z)]'}{[z]'} \prec \frac{1+z}{1-z}.$$

Since $\frac{1+z}{1-z}$ is convex and univalent, we can apply a lemma (see Brown [5], p. 192) to obtain

$$\frac{zp(z)}{z} \prec \frac{1+z}{1-z},$$

from which it follows that

$$p(z) \prec \frac{1+z}{1-z}.$$

Hence, by (2.1) $p \in P$ and $P' \subset P$.

Now let us establish a criterion for a given function to belong to P. By (2.1) $q \in P$ if and only if $q(z) \prec \frac{1+z}{1-z}$. Since $\frac{1+z}{1-z}$ is univalent, then $q \in P$ if and only if $q(z) \neq \frac{1 + e^{i\theta}}{1 - e^{i\theta}}$, for $0 < \theta < 2\pi$ and $|z| < 1$. That is,

$$q \in P \text{ if and only if } (1 - e^{i\theta})q(z) - (1 + e^{i\theta}) \neq 0, \tag{2.3}$$

for $0 < \theta < 2\pi$, $|z| < 1$.

We can express (2.3) in terms of convolutions. Let f and g be analytic in the unit disk D. Recall that if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$, then the convolution (or Hadamard product) of f and g, denoted by $f*g$, is

$$f*g = \sum_{k=0}^{\infty} a_k b_k z^k.$$

Thus, $(1 - e^{i\theta})q(z) - (1 + e^{i\theta})$ can be written as

$$(1 - e^{i\theta})\left[\frac{1}{1-z} * q(z)\right] - (1 + e^{i\theta}) * q(z)$$
$$= \left(\frac{1 - e^{i\theta}}{1 - z} - (1 + e^{i\theta})\right) * q(z).$$

Let $h_\theta(z)$ be defined by

$$h_\theta(z) = -\frac{1}{2e^{i\theta}}\left[\frac{1 - e^{i\theta}}{1 - z} - (1 + e^{i\theta})\right].$$

Then it follows that $h_\theta(0) = 1$ and for $0 < \theta < 2\pi$, $|z| < 1$, $q \in P$ if and only if $h_\theta(z) * q(z) \neq 0$. (2.4)

3. THE MAIN RESULT.

We define a δ-neighborhood of p for $p \in P$.

DEFINITION. For any $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ in P and $\delta \geq 0$, the <u>δ-neighborhood of p</u>, denoted by $N_\delta(p)$, is

$$N_\delta(p) = \left\{ q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k \;\middle|\; \sum_{k=1}^{\infty} |p_k - q_k| \leq \delta \right\}.$$

Our main result is the following theorem.

THEOREM. If $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ belongs to P', then $N_\delta(p) \subset P$, where $\delta = 2 \ln 2 - 1 \simeq .3862944$. This result is sharp.

We need several lemmas.

LEMMA 1. If $p \in P'$, then $z(p*h_\theta)$ is univalent for each $0 < \theta < 2\pi$.

PROOF. Fix $0 < \theta < 2\pi$. Then

$$[z(p*h_\theta)]' = \left[\frac{-z}{2e^{i\theta}}\left((1 - e^{i\theta})p(z) - (1 + e^{i\theta})\right)\right]'$$
$$= -\frac{1}{2}\left[zp(z) - \frac{1 + e^{i\theta}}{1 - e^{i\theta}}\right]' \frac{1 - e^{i\theta}}{e^{i\theta}}$$
$$= -\frac{1}{2}\left[(zp(z))' - \frac{1 + e^{i\theta}}{1 - e^{i\theta}}\right](1 - e^{i\theta})\, e^{-i\theta}. \tag{3.1}$$

By definition of P', the range of $(zp(z))'$ for $|z| < 1$ lies in $\operatorname{Re}(z) > 0$ and that of $\frac{1 + e^{i\theta}}{1 - e^{i\theta}}$ lies on the imaginary axis. Thus, we can choose α so that

$$\operatorname{Re}\{e^{i\alpha}[z(p*h_\theta)(z)]'\} > 0$$

for $|z| < 1$, namely $\alpha = \arg\{-(1 - e^{i\theta})^{-1}e^{i\theta}\}$. By the Noshiro-Warschawski Theorem (Duren [6], p. 47), $z(p*h_\theta)$ is univalent for each θ, $0 < \theta < 2\pi$.

LEMMA 2. If $p \in P'$, then $\left|[z(p*h_\theta)]'\right| > \frac{1 - r}{1 + r}$ for $|z| = r < 1$, $0 < \theta < 2\pi$.

PROOF. Using expression (3.1) for $|z(p*h_\theta)|'$, we define $F(w) = e^{-i\theta}(1 - e^{i\theta})\left(\frac{1 + e^{i\theta}}{1 - e^{i\theta}} - w\right)$, where $w = \frac{1 + re^{it}}{1 - re^{it}}$, $0 \leq t \leq 2\pi$. Now $F(w)$ may be rewritten as

$$F(w) = e^{-i\theta}\{(1 + e^{i\theta}) - (1 - e^{i\theta})w\},\ 0 < \theta < 2\pi.$$

Thus,

$$\left|F(w)\right| = |1 + w|\left|\frac{1 - w}{1 + w} + e^{i\theta}\right|$$

$$= |1 + w||e^{i\theta} - re^{it}|$$
$$= |1 + w||1 - re^{i(t-\theta)}|$$
$$\geq (1 - r)|1 + w|.$$

Since $|1 + w| = \left|1 + \frac{1 + re^{it}}{1 - re^{it}}\right| = \left|\frac{2}{1 - re^{it}}\right| \geq \frac{2}{1 + r}$, it is clear that

$$|F(w)| \geq 2\,\frac{1 - r}{1 + r}.$$

Since $p \in P'$ and (3.1) holds, by letting $w = [zp(z)]'$ we get the desired inequality. That is

$$\left|[z(p*h_\theta)]'\right| \geq \frac{1 - r}{1 + r}.$$

The lemma is proved.

LEMMA 3. If $p \in P'$, then $|p*h_\theta| \geq \delta$, where $\delta = \int_0^1 \frac{1 - t}{1 + t}\,dt = 2 \ln 2 - 1$.

PROOF. Let $p \in P'$. Then by Lemma 1, $z(p*h_\theta)$ is univalent. For fixed $0 < r < 1$, choose z_0 with $|z_0| = r$ such that

$$\min_{|z| = r}|z(p*h_\theta)| = |z_0(p*h_\theta)(z_0)|.$$

Since $z(p*h_\theta)$ is univalent, the preimage L of the line segment from 0 to $z_0\left[(p*h_\theta)(z_0)\right]$ is an arc inside $|z| \leq r$. Hence, for $|z| \leq r$ we have

$$|z(p*h_\theta)| \geq |z_0(p*h_\theta)|$$
$$= \int_L |[z(p*h_\theta)]'||dz|$$
$$\geq \int_0^r |[z(p*h_\theta)]'||dz|.$$

Accordingly, we apply Lemma 2 to get

$$|[p*h_\theta](z)| \geq \frac{1}{r}\int_0^r |[z(p*h_\theta)]'||dz|$$
$$\geq \frac{1}{r}\int_0^r \frac{1 - t}{1 + t}\,dt$$
$$= \frac{2}{r}\ln(1 + r) - 1.$$

The function $g(r) = \frac{2}{r}\ln(1 + r) - 1$ is decreasing for $r > 0$ if $g'(r) = \frac{-2}{r^2}\ln(1 + r) + \frac{2}{r(1 + r)} < 0$. It is not difficult to show that $r - (1 + r)\ln(1 + r) \leq 0$ for $r \geq 0$, from which it follows that $g'(r) < 0$ for $r > 0$. Hence

$$|p*h_\theta| \geq 2 \ln 2 - 1.$$

This completes the proof of Lemma 3. Now we may prove the theorem.

PROOF (OF THEOREM). Let $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \in P'$ and let δ be as in Lemma 3. We want to show that every $q \in N_\delta(p)$ belongs to P, where $q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k$ is an arbitrary but fixed function in $N_\delta(p)$. Hence, $\sum_{k=1}^{\infty}|p_k - q_k| \leq \delta$. Observe that

$$|h_\theta * q| = |(h_\theta * p) + h_\theta*(q - p)|$$

$$\geqslant |h_\theta * p| - |h_\theta * (q - p)|$$
$$\geqslant \delta - \left| \sum_{k=1}^{\infty} \frac{1 - e^{i\theta}}{2} (q_k - p_k) z^k \right|$$
$$> \delta - \sum_{k=1}^{\infty} |q_k - p_k| \geqslant \delta - \delta = 0.$$

Therefore, $h_\theta * q \neq 0$ for $|z| < 1$. By (2.4), it follows that $q \in P$. Consequently, $N_\delta(p) \subset P$.

Now we prove that the result is sharp. Let $p(z)$ be defined by $(zp(z))' = \frac{1 + z}{1 - z}$. Then $p(z) = -1 - \frac{2}{z} \ln (1 - z)$. Now let $q(z) = p(z) + \delta z = -1 - \frac{2}{z} \ln (1 - z) + \delta z$. Clearly, $q \in N_\delta(p)$. However, as $z \to -1$, then $q(z) \to -1 + 2 \ln 2 - \delta = q(-1)$. Therefore, if $\delta > 2 \ln 2 - 1$, then $q(-1) < 0$ and consequently $\mathrm{Re}\, q(z) < 0$ for z near -1. This contradicts $\mathrm{Re}\, q(z) > 0$ for $|z| < 1$. This completes the proof of the theorem.

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