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ON THE EXISTENCE OF THE SOLUTION OF BURGERS' EQUATION FOR $n \leq 4$

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ABSTRACT. In this paper a proof of the existence of the solution of Burgers' equation for $n \leq 4$ is presented. The technique used is shown to be valid for equations with more general types of nonlinearities than is present in Burgers' equation.

KEY WORDS. Burgers' equation, variational methods, quadratic nonlinearities.

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1. INTRODUCTION

Burgers' equation has been used to study a number of physically important phenomena, including shock waves, acoustic transmission and traffic flow. The reader is referred to Fletcher [1] for some of the phenomena that can be modelled, exactly or approximately, by Burgers' equations. Besides its importance in understanding convection-diffusion phenomena, Burgers' equation can be used, especially for computational purposes, as a precursor of the Navier-Stokes equations for fluid flow problems.

In spite the fact that the numerical solution of bergers' equation has received a fair amount of attention (see e.g. Arminjon and Beauchamp [2], Caldwell and Wanless [3], Maday and Quarteroni [4], Caldwell and Smith [5], Fletcher [6] and Saunders et. al. [7]), it seems to draw little theoretical interest. Actully, some of the importance of Burgers' equation stems from the fact that it is one of the few nonlinear equations with known exact solutions in low dimensions. In this direction, the Cole-Hopf transform has been a major tool for finding exact solutions of Burgers' equation in 1 and 2 dimensions (see Fletcher [1]). Benton and Platzman [8] give a table of the known solutions of Burgers' equation.

The goal of this paper is to establish the existence of the solution of the steady Burgers' equation for $n \leq 4$. To the author's knowledge, no such result seems to have been published.

2. THE PROBLEM

Let $\Omega \subset \mathbb{R}^n$ $(n \leq 4)$ be a bounded domain with piecewise C^1 boundary, and consider the n-dimensional Burgers' equation

$$(\mathbf{u}.\nabla)\mathbf{u} = \Delta\mathbf{u} + \mathbf{F} \tag{2.1}$$

where $\mathbf{u}:\Omega \to \mathbb{R}^n$; $\mathbf{F}:\Omega \to \mathbb{R}^n$ (or more generally $\mathbf{F} \in H_0^{-1}(\Omega)^n$),
Δ denotes the Laplacian and for $\mathbf{u}=(u_1,\ldots,u_n)$

$$(\mathbf{u}.\nabla) = \sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j}$$

Equations (2.1) are solved subject to the boundary conditions $\mathbf{u}\big|_\Gamma = 0$ where Γ is the boundary of Ω.

The scalar version of the problem is

$$\Big(\sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j} \Big) u_i = \Delta u_i + F_i \qquad 1 \leq i \leq n \tag{2.2i}$$

$$u_i \big|_\Gamma = 0 \qquad 1 \leq i \leq n \tag{2.3i}$$

We are interested in a variational form of the problem, and we follow an approach which closely resembles that used for the Navier-Stokes equations (See e.g. Temam [9])

Multiplying equation (2i) by w_i $(1 \leq i \leq n)$, integrating by parts over Ω and adding the resulting equations we obtain

$$\sum_{i,j=1}^{n} \int_\Omega u_j \frac{\partial u_i}{\partial x_j} w_i \, dx = - \int_\Omega \nabla\mathbf{u}.\nabla\mathbf{w} \, dx + \int_\Omega \mathbf{F}.\mathbf{w} \, dx \tag{2.4}$$

where $\mathbf{w} = (w_1,\ldots,w_n)$, $\nabla\mathbf{u}.\nabla\mathbf{w} = \sum_{i=1}^{n} \nabla u_i.\nabla w_i$ and $\mathbf{F}.\mathbf{w} = \sum_{i=1}^{n} F_i w_i$.

Now define

$$a(\mathbf{u},\mathbf{w}) = \int_\Omega \nabla\mathbf{u}.\nabla\mathbf{w} \, dx \tag{2.5}$$

$$B(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{n} \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i \, dx \tag{2.6}$$

It is clear that a is a bounded symmetric bilinear form on the space $H_0^1(\Omega)^n$, and the fact that a is coercive follows directly from the Poincare inequality. Thus there exists a constant $\alpha>0$ such that

$$a(\mathbf{u},\mathbf{u}) \geq \alpha \, \|\mathbf{u}\|^2 \qquad \text{for all } \mathbf{u} \in H_0^1(\Omega)^n \tag{2.7}$$

The Sobolev imbedding theorem implies that the integrals on the right hand side of (2.6) are finite (this is where the restriction $n \leq 4$ is needed,) and that for some constant $\beta>0$ we have

$$|B(\mathbf{u},\mathbf{v},\mathbf{w})| \leq \beta \, \|\mathbf{u}\| \|\mathbf{v}\| \|\mathbf{w}\| \qquad \text{for all } \mathbf{u},\mathbf{v},\mathbf{w} \in H_0^1(\Omega)^n \tag{2.8}$$

It is obvious that B is trilinear, i.e. it is linear in each of its arguments. The interested reader can find detailed proofs of the above facts in Temam [9] or Girault and Raviart [10]. In (2.7) and (2.8), $\|.\|$ denotes the usual norm on $H_0^1(\Omega)$;

see e.g. Adams [11].

We now define the variational form of the problem as follows

$$\left.\begin{array}{l}\text{Find } \mathbf{u} \in H_0^1(\Omega)^n \text{ such that} \\ \\ a(\mathbf{u},\mathbf{w}) + B(\mathbf{u},\mathbf{u},\mathbf{w}) = \langle \mathbf{F},\mathbf{w}\rangle^* \quad \text{for every } \mathbf{w} \in H_0^1(\Omega)^n\end{array}\right\} \tag{2.9}$$

where now we allow $\mathbf{F}$ to belong to $H_0^{-1}(\Omega)^n$, and $\langle.,.\rangle^*$ denotes the duality bracket between $H_0^1(\Omega)^n$ and $H_0^{-1}(\Omega)^n$.

Observe that the above problem is almost identical in its formulation to the Navier-Stokes equations, but that the latter problem is posed on a different function space (the space of divergence free vector fields,) and that the trilinear form B of the Navier-Stokes equations possesses an antisymmetry property that makes it easy to obtain an a priori estimate on the solution $\mathbf{u}$ (see Temam [9]) The lack of antisymmetry is what makes Burgers' equation different, and restrictions on the size of the forcing term $\mathbf{F}$ must be imposed in order to establish the existence of the solution.

3. EXISTENCE

In this section we establish the existence of the solution of (2.9). In fact existence follows from the following abstract version of the problem:

Let H be a separable Hilbert space, let a be a bounded symmetric bilinear form on H with the property that for some $\alpha>0$

$$a(u,u) \geq \alpha \|u\|^2 \qquad \text{for all } u \in H \tag{3.1}$$

and finally let B be a trilinear form on H such that there exists a constant $\beta>0$ such that

$$|B(u,v,w)| \leq \beta\|u\|\|v\|\|w\| \qquad \text{for all } u,v,w \in H \tag{3.2}$$

Consider the problem

$$\left.\begin{array}{l}\text{Find } u \in H \text{ such that} \\ \\ a(u,w) + B(u,u,w) = \langle F,w\rangle^* \quad \text{for every } w \in H\end{array}\right\} \tag{3.3}$$

where $F \in H^*$ (the dual of H)

we shall prove the existence of a solution of problem (3.3) under the assumption that

$$\|F\|^* < \frac{\alpha^2}{4\beta} \quad \text{where} \tag{3.4}$$

$$\|F\|^* =: \sup\left\{\frac{|\langle F,u\rangle^*|}{\|u\|} : u \in H,\ u \neq 0\right\} \tag{3.5}$$

Let $r =: (\alpha - \sqrt{\alpha^2 - 4\beta\|F\|^*})/2\beta$

Observe that under assumption (3.4) we have

$$\alpha-\beta r>0 \tag{3.6}$$

$$k =: \frac{\beta\|F\|^*}{(\alpha-\beta r)^2} < 1 \tag{3.7}$$

$$\frac{\|F\|^*}{\alpha-\beta r} = r \tag{3.8}$$

LEMMA 1. For a fixed $u\varepsilon H$ with $\|u\| \le r$, and for $\ell \varepsilon H^*$, the problem

$$a(v,w) + B(u,v,w) = \langle \ell,w\rangle^* \quad \text{for every } w\varepsilon H$$

has a unique solution $v \varepsilon H$.

PROOF: The result follows immediately from (3.6) and the Lax-Milgram theorem since for all $v\varepsilon H$

$$a(v,v) + B(u,v,v) \ge (\alpha-\beta\|u\|)\|v\|^2 \ge (\alpha-\beta r)\|v\|^2 \quad \blacksquare$$

Now let

$$D = \left\{ u\varepsilon H \ : \ \|u\| \le r \right\} \tag{3.9}$$

and define $\Phi : D \to H$ by

$$\Phi(u) = v \tag{3.10}$$

where v is the unique solution of the problem

$$a(v,w) + B(u,v,w) = \langle F,w\rangle^* \qquad \text{for every } w\varepsilon H \tag{3.11}$$

REMARK 1 : Observe that $u\varepsilon D$ is a solution of problem (3.3) if and only if u is a fixed point of Φ.

LEMMA 2. Φ maps D into itself.

PROOF: Choosing $w = v$ in equation (3.11), using (3.1),(3.2) and the fact that $\|u\| \le r$ we obtain

$(\alpha-\beta r)\|v\|^2 \le \|F\|^*\|v\|$, thus by (3.8)

$$\|v\| \le \frac{\|F\|^*}{\alpha-\beta r} = r \quad \blacksquare$$

LEMMA 3. Φ is a contraction on D.

PROOF: For $u\varepsilon D$, define $Au \ \varepsilon \mathcal{L}(H,H^*)$ by

$$\langle Au(v),w\rangle^* = a(v,w) + B(u,v,w) \quad (v,w \ \varepsilon \ H)$$

Observe that

$$\|Au(v)\|^* = \sup \left\{ \frac{|\, a(v,w)+B(u,v,w)\,|}{\|w\|} : w\varepsilon H,\ w\ne 0 \right\} \ge$$

$$\frac{|\, a(v,v)+B(u,v,v)\,|}{\|v\|} \ge \frac{a(v,v) - |\,B(u,v,v)\,|}{\|v\|} \ge (\alpha-\beta r)\|v\|$$

Since $\alpha-\beta r>0$, Au is bounded away from zero, and therefore one-to-one. Lemma 1 states that Au is also onto.

The open mapping theorem implies that Au has a bounded inverse, which we denote by Au^{-1}, and that

$$\|Au^{-1}\| \le \frac{1}{\alpha-\beta r} \tag{3.12}$$

Observe that now ,by definition, $\Phi(u) = Au^{-1}(F)$.

We now show that Φ is a contraction on D. Let u_1, $u_2 \ \varepsilon \ D$ and let $A_i = Au_i$ $(i=1,2)$.

Since $A_2^{-1} - A_1^{-1} = A_2^{-1}(A_1 - A_2)A_1^{-1}$, then by (3.12)

$$\|A_1^{-1} - A_2^{-1}\| \le \frac{1}{(\alpha-\beta r)^2}\|A_1 - A_2\| \tag{3.13}$$

It is easy to verify that

$$\|A_1-A_2\| \le \beta\|u_1-u_2\|. \tag{3.14}$$

Now by (3.13), (3.14) and (3.7)

$$\|\Phi(u_2) - \Phi(u_1)\| = \|A_2^{-1}(F)-A_1^{-1}(F)\| \le \|A_2^{-1} - A_1^{-1}\|\|F\|^* \le$$

$$\frac{\|F\|^*}{(\alpha-\beta r)^2}\|A_1 - A_2\| \le \frac{\beta\|F\|^*}{(\alpha-\beta r)^2}\|u_1-u_2\| = k\|u_1-u_2\|$$

The proof is complete since $k < 1$ by (3.7) ∎

Existence now follows directly from lemma 3 (see remark 1).

THEOREM 4. If $\|F\|^* < \alpha^2/4\beta$, then problem (3.3) has a unique solution u with $\|u\| \le r$. In particular, under the same assumption, the same conclusion is valid for problem (2.9) ∎

Although the above theorem deos not assert the global uniqueness of the solution, one can prove the following result which rules out the existence of other solutions in a certain annulus surrounding D.

THEOREM 5. Under the assumption that $\|F\|^* < \alpha^2/4\beta$, problem (3.3), and hence problem (2.9), has no solutions in the annular region

$$r < \|u\| < r_1, \text{ where } r_1 = (\alpha + \sqrt{\alpha^2 - 4\beta\|F\|^*})/2\beta.$$

PROOF: Let u be a solution of problem (3.3). Choosing $w=u$ in (3.3) we have $a(u,u)+B(u,u,u)=\langle F,u\rangle^*$. Thus $\alpha\|u\|^2 - \beta\|u\|^3 \le \|F\|^*\|u\|$. Hence

$$\beta\|u\|^2 - \alpha\|u\| + \|F\|^* \ge 0 \tag{3.15}$$

Observe that r and r_1 are the roots of the quadratic equation

$\beta\lambda^2 - \alpha\lambda + \|F\|^* = 0$; thus if $r < \|u\| < r_1$, inequality (3.15) cannot hold. ∎

REMARK 2 : It should be observed that the same existence result is valid for quadratic nonlinearities of a much more general nature than the one involved in Burgers' equation. Consider for example a system of the type

$$L_i u_i + Q_i = F_i(x); \; u_i\big|_\Gamma = 0, \quad 1 \le i \le n$$

where $u_i : \Omega \to \mathbf{R}$, $F_i \in H_0^{-1}(\Omega)$, L_i is a linear second order formal differential operator, and $Q_i = Q_i(u_1,\dots,u_n,\nabla u_1,\dots,\nabla u_n)$.

If (a) Each of the operators L_i is strongly elliptic and

(b) Each Q_i is a quadratic form of the variables $Eu =: (u_1,\dots,u_n,\nabla u_1,\dots,\nabla u_n)$ induced by the bilinear form B_i such that for some constants c_i we have

$$\left|\int_\Omega B_i(\mathbf{Eu},\mathbf{Ev})\, w\, dx\right| \le c_i\|\mathbf{u}\|\|\mathbf{v}\|\|w\| \quad (\mathbf{u},\mathbf{v} \in H_0^1(\Omega)^n,\; w \in H_0^1(\Omega)) \tag{3.16}$$

then the same formulation is possible and the results of thoerems 4 and 5 hold when the forcing term is small.

Examples of the above situation include nonliearities which consist of sums involving $u_i u_j$ or $u_i\, \partial u_j/\partial x_k$. In both cases, condition (3.16) follows directly from the Sobolev imbedding theorem; assuming again that $n \le 4$.

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