

**ON POINT - DISSIPATIVE SYSTEMS OF DIFFERENTIAL  
EQUATIONS WITH QUADRATIC NONLINEARITY**

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**ABSTRACT.** The system  $x' = Ax + f(x)$  of nonlinear vector differential equations, where the nonlinear term  $f(x)$  is quadratic with orthogonality property  $x^T f(x) = 0$  for all  $x$ , is point-dissipative if  $u^T A u < 0$  for all nontrivial zeros  $u$  of  $f(x)$ .

**KEY WORDS AND PHRASES.** *Point-dissipative, quadratic nonlinearity, symmetric matrices, commutative but generally non-associative algebra.*

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**I. INTRODUCTION.**

We are concerned with a class of nonlinear vector equations of the form

$$x' = Ax + f(x) \tag{1.1}$$

where the nonlinear term  $f(x)$  is quadratic of the form

$$f(x) = \begin{bmatrix} x^T C_1 x \\ \vdots \\ x^T C_n x \end{bmatrix}$$

The  $n \times n$  matrices  $\{C_i\}$  are symmetric with the orthogonality property

$$x^T f(x) = 0 \tag{1.2}$$

for all  $x$ .

We are interested in investigating the conditions on the  $n \times n$  matrix  $A$  and  $f(x)$  so that the system is point-dissipative, i.e., there is a bounded region which every trajectory of the system eventually enters and remains within.

## II. DEFINITIONS.

For each vector  $\alpha^T = (\alpha_1, \alpha_2, \dots, \alpha_n)$ , we define the matrix  $C(\alpha)$  as follows:

$$C(\alpha) = \sum_{i=1}^n \alpha_i C_i - \frac{A+A^T}{2} \quad (2.1)$$

The mapping  $xQy: R^n \times R^n \rightarrow R^n$ , where

$$xQy = \begin{pmatrix} x^T C_1 y \\ \vdots \\ x^T C_n y \end{pmatrix} \quad (2.2)$$

can be regarded as a commutative multiplication in  $R^n$ . Note that

$$f(x) = xQx$$

$$f(c_1 x) = c_1 xQc_1 x = c_1^2 xQx = c_1^2 f(x)$$

and the quadratic formula

$$f(c_1 u_1 + c_2 u_2 + c_3 u_3) = \sum_{i,j=1}^3 c_i c_j u_i Q u_j \quad (2.3)$$

is true for all vectors  $u_1, u_2, u_3$  and all scalars  $c_1, c_2, c_3$ .

In addition to the standard vector addition and scalar multiplication in  $R^n$ , this multiplication  $xQy$  gives the vector space  $R^n$  an additional structure of a commutative but generally non-associative algebra  $B$ . The algebra  $B$  is determined uniquely by the symmetric  $n \times n$  matrices  $\{C_i\}$ . This algebra has been studied by many specially by Markus [1], Gerber, [2], and Frayman [3].

Some algebraic properties of this algebra  $B$  will be used to investigate the conditions for point-dissipativeness of the system (1.1). We are specially interested in the concepts of nilpotent and idempotent elements of the algebra  $B$ . A nilpotent element  $v \neq 0$  satisfies  $f(v) = vQv = 0$ , while an idempotent element  $v \neq 0$  satisfies  $f(v) = vQv = v$ . It has been proved [3] that in any such algebra  $B$  (with or without the orthogonality property  $x^T(xQx) = 0$  for all  $x$ ) generated by any given  $n$  symmetric matrices  $\{C_i\}$ , there exists at least one of these elements.

In our case, because of the orthogonality property (1.2), there cannot exist an idempotent element in the algebra  $B$ . For, if  $u \neq 0$  is an idempotent, then  $0 = u^T f(u) = u^T (uQ u) = u^T u = \|u\|^2 \neq 0$  gives us a contradiction. Hence, there must exist at least one nilpotent element in the algebra  $B$ . Again by (2.3), a scalar multiple of a nilpotent is also a nilpotent. Hence, the nonlinear quadratic term  $f(x)$  in (1.1) has at least one 1-dimensional subspace of zeros.

As an example of system (1.1) with orthogonality property (1.2), we cite the Lorenz system:

$$x' = Ax + f(x) \quad (2.4)$$

where

$$A = \begin{pmatrix} -a & a & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}, \quad a > 0, r > 0, b > 0$$

$$f(x) = \begin{pmatrix} 0 \\ -xz \\ xy \end{pmatrix}$$

III. **LEMMA 1.** *If there exists an  $\alpha$  so that  $C(\alpha)$  is positive definite, then the system  $x' = Ax + f(x)$  is point-dissipative.*

The condition on  $A$  and  $f(x)$  which guarantees the existence of such an  $\alpha$  is the topic of our main theorem.

**PROOF OF LEMMA 1.** Suppose that there exists a vector  $\alpha$  such that the matrix  $C(\alpha)$  is positive definite. To show that the system (1.1) is point-dissipative, we need to exhibit a bounded region  $G$  so that the (positive) trajectory of each solution of (1.1) eventually enters and remains in  $G$ . We construct a Lyapunov function of the form

$$V(x) = \frac{1}{2}(x - \alpha)^T(x - \alpha)$$

for which

$$\dot{V}(x) = \alpha^T Ax - x^T C(\alpha) x$$

Since the quadratic term  $x^T C(\alpha) x$  dominates the linear term  $-\alpha^T Ax$ , the set

$$S = \{x \mid \dot{V}(x) \geq 0\} \tag{3.1}$$

is bounded. Hence we can choose  $r_0 > 0$ , sufficiently large, so that the level set (sphere)  $V(x) = r_0$  contains in its interior the bounded set  $S$ . We choose the interior of the sphere  $V(x) = r_0$  to be our bounded region  $G$ . Let  $P_0$  be a point outside of  $G$  and  $\Phi(t, P_0)$  be the solution of (1.1) with  $\Phi(0, P_0) = P_0$ . Let  $V(x) = r_1$  be the level set of  $V(x)$  passing through  $P_0$ . Clearly  $r_1 > r_0$ . Let  $H$  be the annular closed region formed by the two concentric spheres  $V(x) = r_1$  and  $V(x) = r_0$ . Since the bounded set  $S$  lies inside the sphere  $V(x) = r_0$ ,  $\dot{V}(x) < 0$  on  $H$ . Therefore,  $V(\Phi(t, P_0))$  is a decreasing function of  $t$  on  $H$ . Hence, the trajectory of  $\Phi(t, P_0)$  must enter the sphere  $V(x) = r_1$  and cannot go outside of the sphere  $V(x) = r_1$  at any time  $t > 0$ . Suppose that the trajectory of  $\Phi(t, P_0)$  cannot enter the region  $G$ . Then it must remain in  $H$  for all time  $t \geq 0$ . It must have a limit point  $P$  in  $H$ . By using standard proof we can show that  $\dot{V}(P) = 0$  which gives us a contradiction as  $\dot{V}(x) < 0$  on  $H$ . Hence, the trajectory of  $\Phi(t, P_0)$  must eventually enter the bounded region  $G$  and cannot go out of  $G$  by the decreasing property of  $V(\Phi(t, P_0))$  and therefore must remain in  $G$ .

IV. **THEOREM.** *For  $n = 2, 3$ , the system  $x' = Ax + f(x)$  is point-dissipative if and only if  $u^T Au < 0$  for all nontrivial zeros  $u$  of  $f(x)$ .*

For  $n = 2$ , the theorem has already been proved by Bose and Reneke [1]. Hence we will give the proof for  $n = 3$ . In order to prove the theorem, all we need to show is that the condition  $u^T Au < 0$  for all nontrivial zeros of  $f(x)$  implies that there exists a vector  $\alpha$  such that the matrix  $C(\alpha)$  is positive definite. Hence, by Lemma 1, the theorem will be proved.

We also need the following definitions and lemmas:

DEFINITION 1. Let  $Z$  be the set of all zeros of  $f(x)$ . That is  $Z$  contains the zero vector and all the nilpotents of the algebra  $B$ .

DEFINITION 2.  $S(u, v)$  is the 2-dimensional subspace of  $R^3$  generated by two linearly independent vectors  $u$  and  $v$ .

DEFINITION 3.  $S(u)$  is the 1-dimensional subspace of  $R^3$  generated by a nontrivial vector  $u$ .

LEMMA 2. If  $u$  is a zero of  $f(x)$ , then  $uQx$  is orthogonal to  $u$  for all  $x$ .

LEMMA 3. If  $u, v$  are two linearly independent zeros of  $f(x)$ , then  $S(u, v) \subset Z$  if and only if  $uQv = 0$ .

PROOF OF LEMMA 2. Suppose that  $u$  be a zero of  $f(x)$ . Then by using the quadratic formula (2.3) and the orthogonality relations  $(u + x)^T f(u + x) = 0$ ,  $(u - x)^T f(u - x) = 0$ , we can show that  $u^T(uQx) = 0$ , for all  $x$ .

PROOF OF LEMMA 3. Let  $u$  and  $v$  be two linearly independent zeros of  $f(x)$ . Suppose that  $uQv = 0$ . Then  $f(c_1u + c_2v) = c_1^2 uQu + 2c_1c_2 uQv + c_2^2 vQv = 0$  implies that  $c_1u + c_2v$  is in  $Z$  for any two scalars  $c_1$  and  $c_2$ . Hence,  $S(u, v) \subset Z$ . Conversely, suppose that  $S(u, v) \subset Z$ . Then  $u + v$  is in  $Z$  and

$$0 = f(u + v) = uQu + 2uQv + vQv = 2 uQv \text{ implies that } uQv = 0.$$

Let  $u_1, u_2, u_3$  be a basis of  $R^3$ , then for any vector  $x = d_1 u_1 + d_2 u_2 + d_3 u_3$ ,

$$x^T C(\alpha) x = \alpha^T f(x) - x^T A x = d^T \hat{C}(\alpha) d \quad (4.1)$$

where  $d^T = (d_1 d_2, d_3)$  and the matrix  $\hat{C}(\alpha) = ((c_{ij}))$  with

$$c_{ij} = \alpha^T (u_i Q u_j) - u_i^T A u_j, \quad i, j = 1, 2, 3,$$

$$c_{ij} = c_{ji}$$

Hence, in order to show that the matrix  $C(\alpha)$  is positive definite for some  $\alpha$ , all we need to show is that the matrix  $\hat{C}(\alpha)$  is positive definite for some  $\alpha$ .

PROOF OF THE THEOREM. That the condition " $u^T A u < 0$  for all nontrivial  $u$  in  $Z$ " is necessary follows from (4.1). Hence we need only to show that it is also sufficient.

The proof of the theorem depends on the nature of the set  $Z$  of all zeros of  $f(x)$ . We need to consider the following cases:

- Case 1.      (a)  $Z$  contains 3 linearly independent vectors with three 2-dimensional subspace of zeros.  
                  (b)  $Z$  contains 3 linearly independent vectors with two 2-dimensional subspace of zeros.  
                  (c)  $Z$  contains 3 linearly independent vectors with one 2-dimensional subspace of zeros.  
                  (d)  $Z$  contains 3 linearly independent vectors with no 2-dimensional subspace of zeros.

- Case 2. (a) Z contains 2 linearly independent vectors with one 2-dimensional subspace of zeros.
- (b) Z contains 2 linearly independent vectors with no 2-dimensional subspace of zeros.

Case 3. Z contains only one linearly independent vector.

Case 1(a) cannot happen. For suppose that  $u_1, u_2, u_3$  be three linearly independent vector in Z so that  $Z = S(u_1, u_2) \cup S(u_1, u_3) \cup S(u_2, u_3)$ . Then by lemma 3

$u_i Q u_j = 0$ , for all  $i, j = 1, 2, 3$ . Hence, for any vector  $x = c_1 u_1 + c_2 u_2 + c_3 u_3$ ,  $f(x) = \sum_{i,j=1}^3 c_i c_j u_i Q u_j = 0$ , implies that  $f(x) = 0$ , for all  $x$ .

Case 1(b) also cannot happen. For suppose that  $u_1, u_2, u_3$  be three linearly independent vectors in Z so that  $Z = S(u_1, u_2) \cup S(u_1, u_3) \cup S(u_3)$ . Then by lemma 3,  $u_i Q u_i = 0$ , for  $i = 1, 2, 3$ ,  $u_1 Q u_2 = 0$ ,  $u_1 Q u_3 = 0$  but  $u_2 Q u_3 \neq 0$ . Now  $f(u_1 + u_2 + u_3) = 2u_2 Q u_3$  and  $(u_1 + u_2 + u_3)^T f(u_1 + u_2 + u_3) = 0$  implies that  $u_1^T (u_2 Q u_3) = 0$ . This implies by lemma 2 that  $u_2 Q u_3$  is orthogonal to each of the basis vector  $u_1, u_2, u_3$  and hence  $u_2 Q u_3 = 0$ , contradicting our hypothesis.

Case 1(c). Let  $u_1, u_2, u_3$  be three linearly independent vectors in Z so that  $Z = S(u_1, u_2) \cup S(u_3)$ . Here  $u_i Q u_i = 0$ ,  $i = 1, 2, 3$ ,  $u_1 Q u_2 = 0$  but  $u_1 Q u_3 \neq 0$ ,  $u_2 Q u_3 \neq 0$ . By hypothesis of the theorem

$$\begin{aligned} (c_1 u_1 + c_2 u_2)^T A (c_1 u_1 + c_2 u_2) &= \sum_{i,j=1}^2 c_i c_j u_i^T A u_j \\ &= (c_1, c_2) \begin{pmatrix} u_1^T A u_1 & u_1^T A u_2 \\ u_1^T A u_2 & u_2^T A u_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} < 0 \end{aligned}$$

for all  $(c_1, c_2) \neq (0, 0)$ . That is

$$\begin{pmatrix} -u_1^T A u_1 & -u_1^T A u_2 \\ -u_1^T A u_2 & -u_2^T A u_2 \end{pmatrix} \text{ is positive definite.}$$

Again  $u_1 Q u_3$  and  $u_2 Q u_3$  must be linearly independent. For suppose that  $c_1(u_1 Q u_3) + c_2(u_2 Q u_3) = 0$ , for some scalars  $c_1$  and  $c_2$ . Taking inner product respectively with  $u_1$  and  $u_2$  and using lemma 2, we get

$$c_2 u_1^T (u_2 Q u_3) = 0$$

$$c_1 u_2^T (u_1 Q u_3) = 0.$$

Now  $u_1^T (u_2 Q u_3) = 0$  implies by lemma 2 that  $u_2 Q u_3$  is orthogonal to each of the basis vector  $u_1, u_2, u_3$  and hence  $u_2 Q u_3 = 0$  contradicting our hypothesis that  $u_2 Q u_3 \neq 0$ .

Therefore  $u_1^T(u_2Qu_3) \neq 0$ , implying that  $c_2 = 0$ . Similarly  $c_1 = 0$ . Hence  $u_1Qu_3$  and  $u_2Qu_3$  are linearly independent. We can choose a vector  $\alpha$  such that

$$\alpha^T (u_1Qu_3) - u_1^T Au_3 = 0$$

$$\alpha^T (u_2Qu_3) - u_2^T Au_3 = 0.$$

For such a choice of  $\alpha$ , the matrix  $\hat{C}(\alpha)$  becomes

$$\hat{C}(\alpha) = \begin{pmatrix} -u_1^T Au_1 & -u_1^T Au_2 & 0 \\ -u_1^T Au_2 & -u_2^T Au_2 & 0 \\ 0 & 0 & -u_3^T Au_3 \end{pmatrix}$$

which is positive definite.

Case 1(d). Let  $u_1, u_2, u_3$  be three linearly independent vectors in  $Z$  so that

$$Z = S(u_1) \cup S(u_2) \cup S(u_3).$$

Here  $u_iQu_j = 0$ , if  $i = j$  and  $u_iQu_j \neq 0$ , if  $i \neq j$ . As in case 1(c), we can show that  $u_1Qu_2, u_1Qu_3, u_2Qu_3$  are linearly independent. Hence we can choose a vector  $\alpha$  such that

$$c_{12} = \alpha^T (u_1Qu_2) - u_1^T Au_2 = 0$$

$$c_{13} = \alpha^T (u_1Qu_3) - u_1^T Au_3 = 0$$

$$c_{23} = \alpha^T (u_2Qu_3) - u_2^T Au_3 = 0.$$

For such a choice of  $\alpha$ , the matrix  $\hat{C}(\alpha)$  becomes

$$\hat{C}(\alpha) = \begin{pmatrix} -u_1^T Au_1 & 0 & 0 \\ 0 & -u_2^T Au_2 & 0 \\ 0 & 0 & -u_3^T Au_3 \end{pmatrix}$$

which is positive definite.

Case 2(a). Let  $u_1, u_2$  be two linearly independent vectors in  $Z$  such that

$Z = S(u_1, u_2)$ . We can assume that  $u_1$  and  $u_2$  are two unit vectors orthogonal to each other. Let  $u_3$  be a unit vector such that  $u_1, u_2, u_3$  form an orthonormal basis of  $\mathbb{R}^3$ . Here,

$$u_1Qu_1 = u_1Qu_2 = u_2Qu_2 = 0, \quad u_3Qu_3 \neq 0$$

Since  $u_1Qu_3$  is orthogonal to  $u_1$  and  $u_3Qu_3$  is orthogonal to  $u_3$ , we can write

$$u_1Qu_3 = t_1u_2 + t_2u_3, \quad u_3Qu_3 = p_1u_1 + p_2u_2.$$

Using the orthogonality property  $(u_1 + u_3)^T f(u_1 + u_3) = 0$ , we can show that

$p_1 = -2t_2$ . Hence,  $u_3Qu_3 = -2t_2u_1 + p_2u_2$ ,  $(t_2, p_2) \neq (0, 0)$ . Similarly we can show that

$u_2Qu_3 = -t_1u_1 - \frac{1}{2}p_2u_3$ . Now, in this case  $t_1 = 0$ . For  $t_1 \neq 0$  implies that

$$f\left(-\frac{p_2}{2t_1}u_1 - \frac{t_2}{t_1}u_2 + u_3\right) = 0. \text{ Since } -\frac{p_2}{2t_1}u_1 - \frac{t_2}{t_1}u_2 + u_3 \text{ is not in } Z, \text{ we get a}$$

contradiction. Hence,  $u_1 Q u_3 = t_2 u_3$ ,  $u_2 Q u_3 = -\frac{1}{2} p_2 u_3$ ,  $u_3 Q u_3 = -2t_2 u_1 + p_2 u_2$ . As in case 1(c),

$$\begin{pmatrix} -u_1^T A u_1 & -u_1^T A u_2 \\ -u_1^T A u_2 & -u_2^T A u_2 \end{pmatrix}$$

is positive definite. Taking  $\alpha = -\frac{1}{2} r t_2 u_1 + r p_2 u_2$ , where  $r > 0$ , to be chosen suitably,

the matrix  $\hat{C}(\alpha)$  becomes

$$\hat{C}(\alpha) = \begin{pmatrix} -u_1^T A u_1 & -u_1^T A u_2 & -u_1^T A u_3 \\ -u_1^T A u_2 & -u_2^T A u_2 & -u_2^T A u_3 \\ -u_1^T A u_3 & -u_2^T A u_3 & r(t_2^2 + p_2^2) - u_3^T A u_3 \end{pmatrix}$$

Here,  $\det \hat{C}(\alpha) = r(t_2^2 + p_2^2) \begin{vmatrix} -u_1^T A u_1 & -u_1^T A u_2 \\ -u_1^T A u_2 & -u_2^T A u_2 \end{vmatrix} + \delta$  where  $\delta$  is a constant

(independent of  $r$ ). Clearly we can choose  $r > 0$ , sufficiently large, to make  $r(t_2^2 + p_2^2) - u_3^T A u_3 > 0$  and  $\det \hat{C}(\alpha) > 0$ . In other words we can choose a vector  $\alpha$  such that  $\hat{C}(\alpha)$  is positive definite.

Case 2(b). Let  $u_1, u_2$  be two linearly independent unit vectors in  $Z$  so that  $Z = S(u_1) \cup S(u_2)$ . Let  $u_3$  be a unit vector orthogonal to  $S(u_1, u_2)$ . Then  $u_1, u_2, u_3$  form a basis of  $R^3$  with  $u_3^T u_1 = 0$ ,  $u_3^T u_2 = 0$ . Here,  $u_1 Q u_2 \neq 0$  and  $u_3 Q u_3 \neq 0$ . As in previous cases we can show using lemma 2 and the orthogonality property of  $f(x)$  that

$$u_1 Q u_2 = s_2 u_3, \quad s_2 \neq 0, \quad u_1 Q u_3 = -(t_1 u_1^T u_2) u_1 + t_1 u_2 + t_2 u_3,$$

$$u_3 Q u_3 = -(2t_2 + q_2 u_1^T u_2) u_1 + q_2 u_2, \quad \text{and}$$

$$u_2 Q u_3 = -\left(t_1 + \frac{s_2}{1 - (u_1^T u_2)^2}\right) u_1 + \left(t_1 + \frac{s_2}{1 - (u_1^T u_2)^2}\right) (u_1^T u_2) u_2 + \frac{1}{2} \left\{ 2t_2 u_1^T q_2 \left( 1 - (u_1^T u_2)^2 \right) \right\} u_3.$$

Now in this case  $t_2 = 0$  implies  $t_1 = 0$ . For, if  $t_2 = 0$ , then  $f\left(\frac{q_2}{2} u_1 - t_1 u_3\right) = 0$  implies that  $t_1 = 0$ . Hence  $t_1 \neq 0$  implies that  $t_2 \neq 0$ .

In order to prove case 2(b), we also need the following two results:

$$(i) \text{ If } t_1 \neq 0, \text{ then } t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} = 0$$

$$(ii) \text{ If } t_1 = 0, \text{ then } 2t_2 u_1^T u_2 - q_2 \left( 1 - (u_1^T u_2)^2 \right) \neq 0$$

To prove result (i), suppose that  $t_1 \neq 0$ . We need to show that the vectors  $u_1 Q u_2, u_1 Q u_3,$

$u_2Qu_3$  are linearly dependent. Suppose that they are linearly independent.

Then  $u_3Qu_3 = c_1(u_1Qu_2) + c_2(u_1Qu_3) + c_3(u_2Qu_3)$  for some  $(c_1, c_2, c_3) \neq (0, 0, 0)$ . Now

$$f\left(\frac{1}{2}c_2u_1 + \frac{1}{2}c_3u_2 - u_3\right) = \left(\frac{1}{2}c_2c_3 + c_1\right)(u_1Qu_2) = \left(\frac{1}{2}c_2c_3 + c_1\right)s_2u_3$$

Since  $\left(\frac{1}{2}c_2u_1 + \frac{1}{2}c_3u_2 - u_3\right)^T f\left(\frac{1}{2}c_2u_1 + \frac{1}{2}c_3u_2 - u_3\right) = 0$ , we have  $\frac{1}{2}c_2c_3 + c_1 = 0$ . This in

turn implies that  $f\left(\frac{1}{2}c_2u_1 + \frac{1}{2}c_3u_2 - u_3\right) = 0$  giving us a contradiction. Hence

$c_1(u_1Qu_2) + c_2(u_1Qu_3) + c_3(u_2Qu_3) = 0$ , for some  $(c_1, c_2, c_3) \neq (0, 0, 0)$ . That is

$$\begin{aligned} & - \left\{ c_2 t_1 u_1^T u_2 + \left( t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \right) c_3 \right\} u_1 + \left\{ c_2 t_1 + \left( t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \right) (u_1^T u_2) c_3 \right\} u_2 \\ & + \left[ c_1 s_2 + c_2 t_2 + \frac{1}{2} c_3 \left\{ 2t_2 u_1^T u_2 - q_2 (1 - (u_1^T u_2)^2) \right\} \right] u_3 = 0 \end{aligned}$$

That is  $(c_1, c_2, c_3) \neq (0, 0, 0)$  must be a solution of the linear system

$$\begin{aligned} c_2 t_1 \left( u_1^T u_2 \right) + \left( t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \right) c_3 &= 0 \\ c_2 t_1 + \left( t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \right) (u_1^T u_2) c_3 &= 0 \\ c_1 s_2 + c_2 t_2 + \frac{1}{2} \left\{ 2t_2 u_1^T u_2 - q_2 (1 - (u_1^T u_2)^2) \right\} c_3 &= 0 \end{aligned}$$

$$\text{Now } \begin{vmatrix} t_1 u_1^T u_2 & t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \\ t_1 & \left( t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \right) (u_1^T u_2) \end{vmatrix} = t_1 \left( t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \right) \left( (u_1^T u_2)^2 - 1 \right).$$

Since  $u_1$  and  $u_2$  are two linearly independent unit vectors,  $|u_1^T u_2| < 1$  and therefore

$$(u_1^T u_2)^2 - 1 \neq 0. \text{ If } t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \neq 0, \text{ then } c_2 = c_3 = 0. \text{ This in turn implies that } c_1 = 0$$

contradicting our hypothesis that  $(c_1, c_2, c_3) \neq (0, 0, 0)$ . Hence  $t_1 \neq 0$  implies that

$$t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} = 0.$$

To prove result (ii), suppose that  $t_1 = 0$ . If  $2t_2(u_1^T u_2) - q_2(1 - (u_1^T u_2)^2) = 0$ , then

$$(2t_2 + q_2(u_1^T u_2))(u_1^T u_2) = q_2, u_1Qu_2 = s_2u_3, s_2 \neq 0, u_1Qu_3 = t_2u_3,$$

$$u_3Qu_3 = \frac{-q_2}{u_1^T u_2} u_1 + q_2 u_2 \text{ (assuming } u_1^T u_2 \neq 0) \text{ and } u_2Qu_3 = \frac{s_2}{1 - (u_1^T u_2)^2} \left\{ -u_1 + (u_1^T u_2) u_2 \right\}$$

$$\text{and } f \left( -\frac{q_2}{u_1^T u_2} u_2 + \frac{2s_2}{1 - (u_1^T u_2)^2} u_3 \right) = 0. \text{ Since } s_2 \neq 0, \text{ this implies a contradiction.}$$

Therefore  $t_1 = 0$  implies that  $2t_2(u_1^T u_2) - q_2(1 - (u_1^T u_2)^2) \neq 0$ . In case  $u_1^T u_2 = 0$ , we can show that  $q_2 \neq 0$ .

To prove case 2(b), we will consider the following two subcases:

- (g)  $t_1 \neq 0$ , and
- (h)  $t_1 = 0$ .

Consider the subcase (g) first. We have  $t_1 \neq 0$ , then by result (i)  $t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} = 0$ .

For this subcase  $u_1Qu_2 = s_2u_3, s_2 \neq 0, u_1Qu_3 = -(t_1u_1^T u_2) u_1 + t_1u_2 + t_2u_3,$

$$u_2Qu_3 = \{t_2(u_1^T u_2) - \frac{1}{2} q_2 (1 - (u_1^T u_2)^2)\} u_3, u_3Qu_3 = -(2t_2 + q_2 u_1^T u_2) u_1 + q_2 u_2.$$

Taking  $\alpha = k_1u_1 + k_2u_2 + k_3u_3$  the entries  $c_{ij}$  of the matrix  $\hat{C}(\alpha)$  becomes,

$$c_{11} = -u_1^T Au_1, c_{22} = -u_2^T Au_2$$

$$c_{12} = \alpha^T u_1Qu_2 - u_1^T Au_2 = s_2k_3 - u_1^T Au_2$$

$$c_{13} = \alpha^T u_1Qu_3 - u_1^T Au_3 = t_1(1 - (u_1^T u_2)^2) k_2 + t_2k_3 - u_1^T Au_3$$

$$c_{23} = \alpha^T u_2Qu_3 - u_2^T Au_3 = \{t_2u_1^T u_2 - \frac{1}{2} (1 - (u_1^T u_2)^2) q_2\} k_3 - u_2^T Au_3$$

$$c_{33} = \alpha^T u_3Qu_3 - u_3^T Au_3 = -2t_2k_1 - \{2t_2u_1^T u_2 - q_2 (1 - (u_1^T u_2)^2)\} k_2 - u_3^T Au_3$$

We can choose  $k_3$  so that  $c_{12} = \alpha^T u_1^T Qu_2 - u_1^T Au_2 = 0$ . For this  $k_3$

$c_{23} = \alpha^T u_2Qu_3 - u_2^T Au_3 = \text{constant} = \delta$  (say). After choosing  $k_3$ , we can now choose  $k_2$

so that  $c_{13} = \alpha^T u_1Qu_3 - u_1^T Au_3 = 0$ . After choosing  $k_2$  and  $k_3$  in this way, we now choose

$k_1 = -\frac{1}{2} t_2 r$ , where  $r > 0$  to be chosen suitably. For such a choice of  $\alpha, c_{33} = \alpha^T u_3Qu_3 - u_3^T$

$Au_3 = t_2^2 r + a$  where  $a$  is a constant independent of  $r$  and the matrix  $\hat{C}(\alpha)$  becomes

$$\hat{C}(\alpha) = \begin{pmatrix} -u_1^T Au_1 & 0 & 0 \\ 0 & u_1^T Au_2 & \delta \\ 0 & \delta & t_2^2 r + a \end{pmatrix}$$

Clearly we can choose  $r > 0$ , sufficiently large to make  $\hat{C}(\alpha)$  positive definite. The subcase (h) can be similarly disposed of, using the fact that  $t_1 = 0$  implies

$$2t_2(u_1^T u_2) - q_2(1 - (u_1^T u_2)^2) \neq 0.$$

Case 3. Let  $u$  be a unit vector in  $Z$  so that  $Z = S(u)$ . Let  $u, v, w$  be an orthonormal basis of  $\mathbb{R}^3$ . By our assumption  $vQv \neq 0$  and  $wQw \neq 0$ . Using lemma 2 and the orthogonality property (1.2), we can write

$$\begin{aligned} uQv &= s_1v + s_2w & uQw &= t_1v + t_2w \\ vQv &= -2s_1u + pw & wQw &= -2t_2u + qw \end{aligned}$$

$$vQw = -(t_1 + s_2)u - \frac{1}{2}pv - \frac{1}{2}qw$$

We will solve this case by considering three subcases:

Subcase (a):  $D = \begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix}$  is of rank 2

Subcase (b):  $D$  is of rank 1

Subcase (c):  $D$  is of rank 0

We also need the following two results (i) and (ii):

(i)  $t_2 = 0$  implies  $t_1 = 0$

(ii)  $s_1 = 0$  implies  $s_2 = 0$

The result (i) can be proved as in case 2(b). For the result (ii), suppose that

$s_1 = 0$  and  $s_2 \neq 0$ . Then  $f(\frac{1}{2}pv - s_2v) = 0$  implies a contradiction. Hence,  $s_1 = 0$  implies  $s_2 = 0$ .

Now consider the subcase (a). The matrix  $D$  is non-singular. This implies by (i) and (ii) that  $s_1t_2 \neq 0$ , otherwise we would get a row of zeros. We will like to show that the quadratic form  $x^TDx \neq 0$  for any  $x \neq 0$ . Suppose that there exists  $x^T = (x_1, x_2) \neq (0, 0)$  such that  $x^TDx = 0$ . Since  $s_1t_2 \neq 0$ , it follows that  $x_1x_2 \neq 0$ .

Since  $D$  is non-singular, the transpose  $D^T = \begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix}$  is also non-singular and

$$D^T x = \begin{pmatrix} s_1x_1 + t_1x_2 \\ s_2x_1 + t_2x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Without loss of generality, suppose that  $s_2x_1 + t_2x_2 \neq 0$ . Then for any scalar

$$c f(cu + x_1v + x_2w) = \{2c(s_2x_1 + t_2x_2) + x_1(x_1p - x_2q)\} \left( -\frac{x_2}{x_1}v + w \right)$$

Since  $s_2x_1 + t_2x_2 \neq 0$ , we can choose the scalar  $c$ , to make  $f(cu + x_1v + x_2w) = 0$  contradicting the fact that  $x_1x_2 \neq 0$ . Hence  $x^TDx \neq 0$  for any  $x \neq 0$ . Therefore by continuity,

$$x^TDx = x^T \begin{pmatrix} s_1 & \frac{t_1 + s_2}{2} \\ \frac{t_1 + s_2}{2} & t_2 \end{pmatrix} x$$

is either positive definite or negative definite. In either case

$$s_1t_2 - \frac{1}{4}(t_1 + s_2)^2 > 0 \tag{4.2}$$

(4.2) also implies that  $s_1$  and  $t_2$  are of the same sign.

Taking  $\alpha = k_1u + k_2v + k_3w$ , the entries  $c_{ij}$  of the matrix  $\hat{C}(\alpha)$  becomes

$$c_{11} = \alpha^T u Q u - u^T A u = -u^T A u$$

$$c_{12} = \alpha^T u Q v - u^T A v = s_1k_2 + s_2k_3 - u^T A v$$

$$\begin{aligned} c_{13} &= \alpha^T u Q w - u^T A w = t_1 k_2 + t_2 k_3 - u^T A w \\ c_{22} &= \alpha^T v Q v - v^T A v = -2s_1 k_1 + p k_3 - v^T A v \\ c_{33} &= \alpha^T w Q w - w^T A w = -2t_2 k_1 + q k_2 - w^T A w \end{aligned}$$

$$c_{23} = \alpha^T v Q w - v^T A w = -(t_1 + s_2) k_1 - \frac{1}{2} p k_2 - \frac{1}{2} q k_2 - v^T A w$$

Since  $D$  is non-singular, we can choose  $k_2$  and  $k_3$  so that  $c_{12} = c_{13} = 0$ . Since  $s_1$  and  $t_2$  are of the same sign, we can choose  $k_1$  with  $|k_1|$  sufficiently large to make  $c_{22} > 0, c_{33} > 0$  and

$$\begin{vmatrix} c_{22} & c_{23} \\ c_{23} & c_{33} \end{vmatrix} = \left\{ 4s_1 t_2 - (t_1 + s_2)^2 \right\} k_1^2 + k_1 d_1 + d_2 > 0$$

where  $d_1$  and  $d_2$  are constants. Hence for such a choice of  $k_1, k_2, k_3$  the matrix  $\hat{C}(\alpha)$  becomes

$$\hat{C}(\alpha) = \begin{pmatrix} -u^T A u & 0 & 0 \\ 0 & c_{22} & c_{23} \\ 0 & c_{23} & c_{33} \end{pmatrix}$$

which is positive definite.

Now consider the subcase (b). Here  $\text{rank } D = 1$ . Without loss of generality we can assume that  $(t_1, t_2) \neq (0, 0)$ . This implies that  $t_2 \neq 0$ , by property (i). Let  $(s_1, s_2) = k(t_1, t_2)$ . This implies that  $k = t_1/t_2$ . For suppose that  $k \neq t_1/t_2$ . Then for any scalar  $c$ ,  $f(cu + t_2v - t_1w) = \{2c(kt_2 - t_1) + t_2p + t_1q\} (t_1v + t_2w)$ . Since  $kt_2 - t_1 \neq 0$ , we can choose the scalar  $c$  so that  $f(cu + t_2v - t_1w) = 0$  implying that  $t_1 = t_2 = 0$  contradicting our assumption. This also implies that  $t_2p + t_1q \neq 0$ . Hence

$$D = \begin{pmatrix} t_1^2/t_2 & t_1 \\ t_1 & t_2 \end{pmatrix}$$

With this  $D$

$$uQv = \frac{t_1^2}{t_2} v + t_1 w$$

$$uQw = t_1 v + t_2 w$$

$$vQv = \frac{-2t_1^2}{t_2} u + pw$$

$$wQw = -2t_2 u + qv$$

$$vQw = -2t_1 u - \frac{1}{2} pv - \frac{1}{2} qw$$

Since  $vQv \neq 0$ , we have  $(t_1, p) \neq (0, 0)$ . Taking  $\alpha = \frac{1}{2} r_2 t_2 u + r_1 q v + r_1 p w$ , where  $r_1 > 0$ ,

$r_2 > 0$  to be chosen suitably, the entries  $c_{ij}$  of the matrix  $\hat{C}(\alpha)$  becomes

$$c_{11} = -u^T A u, c_{12} = \frac{r_1 t_1}{t_2} (t_1 q + t_2 p) - u^T A v, c_{13} = r_1 (t_1 q + t_2 p) - u^T A w$$

$$c_{22} = r_2 t_1^2 + r_1 p^2 - v^T A v, c_{23} = t_1 t_2 r_2 - r_1 p q - v^T A w$$

$$c_{33} = r_2 t_2^2 + r_1 q^2 - w^T A w$$

Now  $c_{11} = -u^T A u > 0$ ,  $\begin{vmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{vmatrix} = r_2 (-u^T A u) t_1^2 + d_1(r_1)$ , where  $d_1(r_1)$  is a quadratic

in  $r_1$  and  $\det \hat{C}(\alpha) = r_2 [(-u^T A u) (t_1 q + t_2 p)^2 r_1 + d_2] + d_3(r_1)$ , where  $d_2$  is a constant and  $d_3(r_1)$  is a cubic polynomial in  $r_1$ . Hence, if  $t_1 \neq 0$ , then we can choose  $r_1 > 0$  large enough to make  $(-u^T A u) (t_1 q + t_2 p)^2 r_1 + d_2 > 0$ . After choosing such an  $r_1 > 0$ , we can

choose  $r_2 > 0$  sufficiently large to make  $\begin{vmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{vmatrix} > 0$  and  $\det \hat{C}(\alpha) > 0$ . In

other words we can choose  $\alpha$  so that  $\hat{C}(\alpha)$  is positive definite.

If  $t_1 = 0$ , then  $\begin{vmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{vmatrix} = (-u^T A u) p^2 r_1 + d_4$ , where  $d_4$  is a constant and

$\det \hat{C}(\alpha) = r_2 t_2^2 [(-u^T A u) p^2 r_1 + d_4] + d_5(r_1)$ , where  $d_5(r_1)$  is a quadratic in  $r_1$ . As before we can choose  $r_1 > 0$  to make

$$(-u^T A u) p^2 r_1 + d_4 > 0$$

and after choosing such an  $r_1 > 0$ , we can choose  $r_2 > 0$  to make  $\det \hat{C}(\alpha) > 0$ . In other words we can choose an  $\alpha$  so that  $\hat{C}(\alpha)$  is positive definite.

Now consider the subcase (c). Here  $\text{rank } D = 0$ , which implies that  $s_1 = s_2 = t_1 = t_2 = 0$ .

Hence  $uQv = 0$ ,  $uQw = 0$ ,  $vQv = pw$ ,  $p \neq 0$ ,  $wQw = qv$ ,  $q \neq 0$ ,  $vQw = -\frac{1}{2}pv - \frac{1}{2}qw$  and  $f(qv + pw) = 0$ .

Since  $pq \neq 0$ , this implies a contradiction. Hence, subcase (c) cannot happen.

This completes the proof.

For an example, the Lorenz system (2.4)

$$x' = Ax + f(x), \quad \text{where}$$

$$A = \begin{pmatrix} -a & a & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}, \quad a > 0, \quad r > 0, \quad b > 0 \quad \text{and} \quad f(x) = \begin{pmatrix} 0 \\ -xz \\ xy \end{pmatrix}$$

is point dissipative. The vectors  $u = (1, 0, 0)$ ,  $v = (0, 1, 0)$ ,  $w = (0, 0, 1)$  are three linearly independent zeros of  $f(x)$  and  $Z = S(u) \cup S(v, w)$ . The condition  $u^T A u < 0$  for all  $u \in Z$  can easily be verified.

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