

BEST APPROXIMATION IN ORLICZ SPACES

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ABSTRACT. Let X be a real Banach space and (Ω, μ) be a finite measure space and ϕ be a strictly increasing convex continuous function on $[0, \infty)$ with $\phi(0) = 0$. The space $L_\phi(\mu, X)$ is the set of all measurable functions f with values in X such that

$$\int_{\Omega} \phi(|c^{-1}f(t)|) d\mu(t) < \infty \text{ for some } c > 0. \text{ One of the main results of this paper is:}$$

"For a closed subspace Y of X , $L_\phi(\mu, Y)$ is proximal in $L_\phi(\mu, X)$ if and only if $L^1(\mu, Y)$ is proximal in $L^1(\mu, X)$ ". As a result if Y is reflexive subspace of X , then $L_\phi(\mu, Y)$ is proximal in $L_\phi(\mu, X)$. Other results on proximality of subspaces of $L_\phi(\mu, X)$ are proved.

1. INTRODUCTION.

Let ϕ be a convex Orlicz function, i.e. ϕ is a continuous, strictly increasing convex function defined on $[0, \infty)$ with $\phi(0) = 0$ and let (Ω, μ) be a finite measure. For a real Banach space X , let

$$L_\phi(\mu, X) = \{\text{measurable function } f: \Omega \rightarrow X: \int_{\Omega} \phi(|c^{-1}f(t)|) d\mu(t) < \infty$$

for some $c > 0$. Define a norm on $L_\phi(\mu, X)$ by

$$\|f\|_\phi = \inf \{c > 0: \int_{\Omega} \phi(|c^{-1}f(t)|) d\mu(t) < 1\}.$$

A subspace Y in a Banach space X is called proximal if for each $x \in X$ there is at least one $y \in Y$ such that $\|x - y\| = d(x, Y) = \inf \{\|x - h\|, h \in Y\}$. The element y is called best approximant of x in Y . Set $P(x, Y) = \{y \in Y: d(x, Y) = \|x - y\|\}$.

In this paper we prove that for a closed subspace Y of a Banach space X , $L_\phi(\mu, Y)$ is proximal in $L_\phi(\mu, X)$ if and only if $L^1(\mu, Y)$ is proximal in $L^1(\mu, X)$. In [1] Deeb and Khalil, have shown the same result for the linear metric space $L_\phi(\mu, X)$ with ϕ modulus function and some Banach space X . As a consequence, if Y

is a reflexive subspace of a Banach space X then $L_\phi(\mu, Y)$ is proximal in $L_\phi(\mu, X)$.

The proximality of some closed subspaces in X are discussed. Throughout this paper Ω will be the unit interval $[0,1]$, ϕ convex, strictly increasing with $\phi(0) = 0$, $\phi(1) = 1$ and X is a Banach space. See Deeb and Khalil [1,2,3], Light and Cheney [4], and Khalil [5] for more details about proximality and related topics.

2. PROXIMALITY IN $L_\phi(\mu, X)$.

LEMMA 2.1. If ϕ is convex, then $L_\phi(\mu, X) \subseteq L^1(\mu, X)$.

PROOF. Let $f \in L_\phi(\mu, X)$, then

$$\int_0^1 \phi(|c^{-1}f(t)|)d\mu(t) < M \text{ for some } c \text{ and some } M$$

By Jensen's Inequality, [6]

$$\phi\left(\int_0^1 |c^{-1}f(t)|d\mu(t)\right) \leq \int_0^1 \phi(|c^{-1}f(t)|)d\mu(t) < M$$

or

$$\phi\left(\int_0^1 |c^{-1}f(t)|d\mu(t)\right) < M.$$

Hence

$$\int_0^1 |c^{-1}f(t)|d\mu(t) < \phi^{-1}(M).$$

Therefore

$$\int_0^1 |f(t)|d\mu(t) < c \phi^{-1}(M) < \infty.$$

Hence $f \in L^1(\mu, X)$.

LEMMA 2.2. Let Y be a subspace of X , then for each $f \in L_\phi(\mu, X)$

$$\text{dist}(f, L_\phi(\mu, Y)) = \inf\{c > 0: \int_0^1 \phi(|c^{-1}\text{dist}(f(t), Y)|)d\mu(t) < 1\}.$$

PROOF. For any $g \in L_\phi(\mu, Y)$ we have,

$$\begin{aligned} \|f-g\|_\phi &= \inf\{c > 0: \int_0^1 (|c^{-1}(f(t) - g(t))|)d\mu(t) < 1\} \\ &> \inf\{c > 0: \int_0^1 \phi(|c^{-1}\text{dist}(f(t), Y)|)d\mu(t) < 1\}. \end{aligned}$$

By taking the infimum over $g \in L_\phi(\mu, Y)$ we get

$$\text{dist}(f, L_\phi(\mu, Y)) \geq \inf\{c > 0: \int_0^1 \phi(|c^{-1}\text{dist}(f(t), Y)|)d\mu(t) < 1\}.$$

Conversely, let $\epsilon > 0$ and let f' be a simple function in $L_\phi(\mu, X)$, such that

$$\|f - f'\|_\phi < \epsilon. \quad \text{Write } f' = \sum_{i=1}^n \chi_{A_i} x_i, \text{ where } x_i \in X \text{ and } \chi_{A_i} \text{ are the characteristic}$$

functions on A_i which are disjoint measurable sets in $[0, 1]$. It is clear that $f' \in L_\phi(\mu, X)$. Select $h_i \in Y$ such that

$$\phi(c^{-1} \|x_i - h_i\|) < \phi[c^{-1} \text{dist}(x_i, Y) + \epsilon], \quad \text{for some } c > 0.$$

Let $g = \sum_{i=1}^n \chi_{A_i} h_i$, then

$$\int_0^1 \phi(\|c^{-1} g(t)\|) d\mu(t) = \sum_{i=1}^n \phi(\|c^{-1} h_i\|) \mu(A_i) < \infty.$$

Hence $g \in L_\phi(\mu, Y)$, then

$$\|f - g\|_\phi = \|f - f' + f' - g\|_\phi < \epsilon + \|f' - g\|_\phi$$

But $\text{dist}(f, L_\phi(\mu, Y)) < \|f - g\|_\phi$

$$\begin{aligned} &< \epsilon + \inf\{c > 0: \int_0^1 \phi(c^{-1} \|f'(t) - g(t)\|) d\mu(t) < 1\} \\ &= \epsilon + \inf\{c > 0: \sum_{i=1}^n \int_0^1 \phi(c^{-1} \|x_i - h_i\|) d\mu(t) < 1\} \\ &= \epsilon + \inf\{c > 0: \sum_{i=1}^n \phi(c^{-1} \|x_i - h_i\|) \mu(A_i) < 1\} \\ &< \epsilon + \inf\{c > 0: \sum_{i=1}^n \phi[c^{-1} \text{dist}(x_i, Y) + \epsilon] \mu(A_i) < 1\} \\ &= \epsilon + \inf\{c > 0: \int_0^1 \phi[c^{-1} \text{dist}(f'(t), Y) + \epsilon] d\mu(t) < 1\} \\ &< \epsilon + \inf\{c > 0: \int_0^1 \phi(c^{-1} \text{dist}(f(t), Y) + \|f(t) - f'(t)\| + \epsilon) d\mu(t) < 1\}. \\ &< \epsilon + \inf\{c > 0: \int_0^1 \phi(c^{-1} \text{dist}(f(t), Y) + 2\epsilon) d\mu(t) < 1\}. \end{aligned}$$

Since ϵ is arbitrary, we have

$$\text{dist}(f, L_\phi(\mu, Y)) < \inf\{c > 0: \int_0^1 \phi(c^{-1} \text{dist}(f(t), Y)) d\mu(t) < 1\}.$$

REMARK 2.1. For $f \in L_\phi(\mu, X)$,

$$\|f\|_\phi = \inf\{c > 0: \int_0^1 \phi\left(\frac{\|f(t)\|}{c}\right) d\mu(t) < 1\} = C_0$$

such that $\int_0^1 \phi\left(\frac{\|f(t)\|}{C_0}\right) d\mu(t) = 1.$

COROLLARY 2.1. Let Y be a closed subspace of X . To an element f of $L_\phi(\mu, X)$, g of $L_\phi(\mu, Y)$ is a best approximant of f in $L_\phi(\mu, Y)$ if and only if $g(t)$ is a best approximant of $f(t)$ in Y .

PROOF. Let $g(t)$ be a best approximant of $f(t)$ in Y . This means that

$$\|f(t) - g(t)\| < \|f(t) - y\| \text{ for all } t \text{ and for all } y \in Y.$$

It follows that for any $h \in L_\phi(\mu, Y)$

$$\|f(t) - g(t)\| < \|f(t) - h(t)\| \text{ for all } t.$$

Since ϕ is increasing, we have

$$\phi(c^{-1} \|f(t) - g(t)\|) < \phi(c^{-1} \|f(t) - h(t)\|) \text{ for any } c > 0.$$

Then

$$\int_0^1 \phi(c^{-1} \|f(t) - g(t)\|) d\mu(t) < \int_0^1 \phi(c^{-1} \|f(t) - h(t)\|) d\mu(t).$$

Therefore

$$\inf \{c > 0: \int_0^1 \phi(c^{-1} \|f(t) - g(t)\|) d\mu(t) < 1\} < \inf \{c > 0: \int_0^1 \phi(c^{-1} \|f(t) - h(t)\|) d\mu(t) \leq 1\}$$

or

$$\|f - g\|_\phi < \|f - h\|_\phi \text{ for all } h \in L_\phi(\mu, Y).$$

Conversely, let g be a best approximant of f in $L_\phi(\mu, Y)$, then $\text{dist}(f, L_\phi(\mu, Y)) = \|f - g\|_\phi$. By Lemma 2.2 and the previous remark, we have

$$\|f - g\|_\phi = \inf \{c > 0: \int_0^1 \phi(c^{-1} \text{dist}(f(t), Y)) d\mu(t) < 1\} = c_0 \text{ such that}$$

$$\int_0^1 \phi\left(\frac{\|f(t) - g(t)\|}{c_0}\right) d\mu(t) = \int_0^1 \phi\left(\frac{\text{dist}(f(t), Y)}{c_0}\right) d\mu(t) = 1.$$

$$\text{Hence } \int_0^1 [\phi(c_0^{-1} \|f(t) - g(t)\|) - \phi(c_0^{-1} \text{dist}(f(t), Y))] d\mu(t) = 0$$

since ϕ is strictly increasing and $\phi(c_0^{-1} \|f(t) - g(t)\|) > \phi(c_0^{-1} \text{dist}(f(t), Y))$ then $\|f(t) - g(t)\| = \text{dist}(f(t), Y)$.

Now we prove the main theorem of this paper.

THEOREM 2.1. Let Y be a closed subspace of X , then the following are equivalent:

(i) $L_\phi(\mu, Y)$ is proximal in $L_\phi(\mu, X)$

(ii) $L^1(\mu, Y)$ is proximal in $L^1(\mu, X)$.

PROOF. (ii) \rightarrow (i). Let $f \in L_\phi(\mu, X)$, then by Lemma 2.1 $f \in L^1(\mu, X)$. By the assumption, there exists $g \in L^1(\mu, Y)$ such that

$$\|f-g\|_1 < \|f-h\|_1 \text{ for every } h \in L^1(\mu, Y).$$

By lemma 2.10 [3], we have

$$\|f(t) - g(t)\| < \|f(t) - y\| \text{ for all } t \text{ and for all } y \in Y.$$

Hence by Corollary 2.1 it follows that g is a best approximant of f in $L_\phi(\mu, Y)$.

Conversely: (i) \rightarrow (ii). Define a map

$$J: L^1(\mu, X) \rightarrow L_\phi(\mu, X) \text{ by } J(f) = \hat{f} \text{ where } \hat{f}(t) = \frac{\phi^{-1}(\|f(t)\|)}{\|f(t)\|} f(t)$$

if $f(t) \neq 0$, and zero otherwise. Then for $c = 1$

$$\begin{aligned} \int_0^1 \phi(\|c^{-1}\hat{f}(t)\|) d\mu(t) &= \int_0^1 \phi(\|\frac{\phi^{-1}(\|f(t)\|)}{\|f(t)\|} f(t)\|) d\mu(t) \\ &= \int_0^1 \phi(\|f(t)\|) d\mu(t) < \infty \end{aligned}$$

for all $f \in L^1(\mu, X)$. Hence $J(f) \in L_\phi(\mu, X)$. Since ϕ is strictly increasing, it follows that J is (1-1). To show that J is onto, let $g \in L_\phi(\mu, X)$, then take

$$f(t) = \frac{\phi(\|g(t)\|)}{\|g(t)\|} g(t)$$

if $g(t) \neq 0$ and zero otherwise. Clearly $f \in L^1(\mu, X)$ and

$$\begin{aligned} J(f) &= \frac{\phi^{-1}(\|f(t)\|)}{\|f(t)\|} f(t) \\ &= \frac{\phi^{-1}(\phi(\|g(t)\|))}{\phi(\|g(t)\|)} \frac{\phi(\|g(t)\|)}{\|g(t)\|} g(t) \\ &= g(t). \end{aligned}$$

Thus J is onto. Now let $f \in L^1(\mu, X)$, then $\hat{f} \in L_\phi(\mu, X)$. By assumption there exists $\hat{g} \in L_\phi(\mu, Y)$ such that

$$\|\hat{f} - \hat{g}\|_{\phi} < \|\hat{f} - \hat{h}\|_{\phi} \text{ for all } \hat{h} \in L_{\phi}(\mu, Y),$$

then by Corollary 2.1 we have

$$\|\hat{f}(t) - \hat{g}(t)\| < \|\hat{f}(t) - y\| \text{ for all } y \in Y \text{ or}$$

$$\left\| f(t) - \frac{\|f(t)\| \phi^{-1}(\|g(t)\|)}{\|g(t)\| \phi^{-1}(\|f(t)\|)} g(t) \right\| < \left\| f(t) - \frac{y \|f(t)\|}{\phi^{-1}(\|f(t)\|)} \right\| \text{ for}$$

all $y \in Y$. Put $w(t) = \frac{\|f(t)\| \phi^{-1}(\|g(t)\|)}{\|g(t)\| \phi^{-1}(\|f(t)\|)} g(t)$.

Using the facts that $\|g(t)\| < 2\|f(t)\|$ since $0 \in Y$ and $\phi^{-1}(2\|f(t)\|) < 2(\phi^{-1}(\|f(t)\|))$ we can show that $w \in L^1(\mu, Y)$ as follows

$$\begin{aligned} \|w(t)\| &= \frac{\|f(t)\| \phi^{-1}(\|g(t)\|)}{\phi^{-1}(\|f(t)\|)} \\ &< \frac{\|f(t)\| \phi^{-1}(2\|f(t)\|)}{\phi^{-1}(\|f(t)\|)} \\ &< \frac{2\|f(t)\| \phi^{-1}(\|f(t)\|)}{\phi^{-1}(\|f(t)\|)} \\ &= 2\|f(t)\|. \end{aligned}$$

Now take any $h \in L^1(\mu, Y)$ then

$$\frac{\phi^{-1}(\|f(t)\|)}{\|f(t)\|} h(t) \in Y \text{ for all } t.$$

Hence

$$\|f(t) - w(t)\| < \left\| f(t) - \frac{\|f(t)\|}{\phi^{-1}(\|f(t)\|)} \frac{\phi^{-1}(\|f(t)\|)}{\|f(t)\|} h(t) \right\|$$

= $\|f(t) - h(t)\|$ for all t and for all $h \in L^1(\mu, Y)$, so $L^1(\mu, Y)$ is proximal in $L^1(\mu, X)$.

As a corollary.

COROLLARY 2.2. If Y is a reflexive subspace of X , then $L_{\phi}(\mu, Y)$ is proximal in $L_{\phi}(\mu, X)$.

PROOF. It follows from the main theorem and Theorem 1.2 in Kahalil [5].

THEOREM 2.2. Let Y be a proximal subspace of X . Then for every simple function $f \in L_{\phi}(\mu, X)$, $P(f, L_{\phi}(\mu, Y))$ is not empty.

PROOF. Let $f = \sum_{i=1}^n \chi_{A_i} x_i$ be a simple function in $L_{\phi}(\mu, X)$, where A_i are disjoint measurable sets in $[0, 1]$. Set $g = \sum_{i=1}^n \chi_{A_i} y_i$, where $y_i \in P(x_i, Y)$. Let h be any

element in $L_\phi(\mu, Y)$, then

$$\begin{aligned}
 \|f-h\|_\phi &= \inf\{c > 0: \int_0^1 \phi(|c^{-1}(f(t)-h(t))|)d\mu(t) < 1\} \\
 &= \inf\{c > 0: \sum_{i=1}^n \int_{A_i} \phi(|c^{-1}(f(t) - h(t))|)d\mu(t) < 1\} \\
 &= \inf\{c > 0: \sum_{i=1}^n \int_{A_i} \phi(|c^{-1}(x_i - h(t))|)d\mu(t) < 1\} \\
 &> \inf\{c > 0: \sum_{i=1}^n \int_{A_i} \phi(|c^{-1}(x_i - y_i)|)d\mu(t) < 1\} \\
 &= \inf\{c > 0: \int_0^1 \phi(|c^{-1}(f(t) - g(t))|)d\mu(t) < 1\} \\
 &= \|f - g\|_\phi.
 \end{aligned}$$

Hence $g \in P(f, L_\phi(\mu, Y))$.

THEOREM 2.3. Let Y be a closed subspace of X . If $L_\phi(\mu, Y)$ is proximal in $L_\phi(\mu, X)$, then Y is proximal in X .

PROOF. From Theorem 2.1, $L_\phi(\mu, Y)$ proximal in $L_\phi(\mu, X)$ implies that $L^1(\mu, Y)$ is proximal in $L^1(\mu, X)$. By Theorem 1.1 [2] this also implies that $L^\infty(\mu, Y)$ is proximal in $L^\infty(\mu, X)$. For $x \in X$, define $f_x: \Omega \rightarrow X$ by $f_x(t) = x$ for all $t \in \Omega$. It is clear that $f_x \in L^\infty(\mu, X)$ for every $x \in X$, so there exists $h \in L^\infty(\mu, Y)$ such that

$$\|f_x - h\|_\infty < \|f_x - w\| \text{ for every } w \in L^\infty(\mu, Y).$$

In particular take $w = f_y$, so

$$\begin{aligned}
 \|f_x - h\|_\infty &< \|f_x - f_y\|_\infty \text{ for every } y \in Y \\
 &= \|x - y\| \text{ for every } y \in Y.
 \end{aligned}$$

$$\begin{aligned}
 \text{But } \|x - h(t)\| &= \|f_x(t) - h(t)\| \\
 &< \|f_x - h\| \\
 &< \|f_x - f_y\| \\
 &= \|x - y\| \text{ for every } y \in Y.
 \end{aligned}$$

Hence every $t \in [0, 1]$ gives a best approximant of x in Y . Therefore Y is proximal in X .

The next theorem needs the following definitions:

DEFINITION 2.1. The subspace Y is called ϕ -summand of X if there is a bounded projection $Q: X \rightarrow Y$ such that

$\phi(\|x\|) = \phi(\|(Q(x))\|) + \phi(\|(I-Q)(x)\|)$ for all $x \in X$. Where I is the identity map on X .

DEFINITION 2.2. The subspace Y is called 1-complemented in X if there is a closed subspace Z in X that $X = Y \dot{+} Z$ and the projection $P: X \rightarrow Z$ is a contractive projection.

THEOREM 2.4. If Y is 1-complemented in X , the $L_\phi(\mu, Y)$ is proximal in $L_\phi(\mu, X)$.

PROOF. Let $X = Y \dot{+} Z$, $P: X \rightarrow Z$ be a contractive projection from X onto Z . Hence $x = (I-P)x + p(x)$, $\|P(x)\| < \|x\|$. For $f \in L_\phi(\mu, X)$, set $f_1 = (I-P) \circ f$, $f_2 = p \circ f$. Let $\check{P}: L_\phi(\mu, X) \rightarrow L_\phi(\mu, Z)$ and

$$p(f) = p \circ f = f_2 \text{ for all } f \in L_\phi(\mu, X).$$

then \check{P} is a contractive projection onto $L_\phi(\mu, Z)$ and $L_\phi(\mu, X) = L_\phi(\mu, Y) \dot{+} L_\phi(\mu, Z)$. Hence $L_\phi(\mu, Y)$ is 1-complemented in $L_\phi(\mu, X)$. By Lemma 1.6 [2] $L_\phi(\mu, Y)$ is proximal in $L_\phi(\mu, X)$.

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