

INVERSES OF MEASURES ON A CLASS OF DISCRETE GROUPS

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ABSTRACT. We examine a class of groups G , having a certain growth condition. We give an estimate for the norm of the inverse of an element in $l_1(G)$ in terms of the spectral radius and the cardinality of the support.

KEY WORDS AND PHRASES. Discrete groups, spectral radius and measure algebra.

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1. INTRODUCTION.

Throughout this work G is a discrete group and $M(G)$ is the usual measure algebra on G : we write δ for the unit of $M(G)$, $\mu * \nu$ for the (convolution) product of two measures $\mu, \nu \in M(G)$.

The purpose of this paper is to investigate the following problems: Let $\mu \in M(G)$ have finite support and the (convolution) inverse of $(\delta - \mu)$ exist in $M(G)$. Is it possible to estimate the norm $\|(\delta - \mu)^{-1}\|$ of it?

One can easily realize that this problem becomes more interesting in the "limit" case, where the support of μ is infinite. One also can realize the connection of this with the general problem of the invertibility in $M(G)$; namely the characterization of the class of Hermitian groups G (see [1]) or the equality of different norms and spectrums in $M(G)$ (see [2], [3] and [4]).

Our work here can be separated into two parts. In the first part we consider the class of groups A and we give an estimate of $\|(\delta - \mu)^{-1}\|$ and of $\|\exp \mu\|$ in terms of the spectral radius $r(\mu)$ and of the cardinality of the support of μ .

In the second part we examine the relation of the class A , with the class of nilpotent groups and [FC] groups (groups having finite conjugacy class) (see [1]). We show that nilpotent groups are A -groups and that the class A is closed under finite extension. We should note that by Gromov's well known result, any finitely generated group G with polynomial growth is a finite extension of a nilpotent group, and so it is an A -group. It is remarkable to note that all known Hermitian groups, as they are

referred to in [1], are A-groups. We shall complete this introduction with some definitions and notations.

We say that G is an A-group if there is a map $\kappa: J \rightarrow N$, where J is the set of all finite subsets of G , and a $\lambda \in N$ such that for any $F \in J$

$$\#(F^n) \leq \binom{\kappa(F)+n-1}{n}^\lambda, \quad n \in N$$

where $F^n = F F \dots F$ (n -times) and $\#F$ is the cardinality of F .

An n -word in the elements of F is any (reduced) word of length n . We denote by $(F^n)'$ a collection of n -words in the elements of F which as a subset of G consists of all distinct elements of F^n . Finally we shall denote by μ^n the convolution product $\mu * \mu \dots * \mu$ (n -times); the spectral radius $r(\mu)$ of μ is the limit $\lim_{n \rightarrow \infty} \|\mu^n\|^{1/n}$.

2. NORMS OF CERTAIN INVERTIBLE MEASURES.

We are going to show the following:

THEOREM 2.1. Let G be an A-group and let μ be an element in $M(G)$, with finite support F and $r(\mu) < 1$. Then $\delta - \mu$ is invertible such that

$$\|(\delta - \mu)^{-1}\| \leq (1 - r(\mu))^{-\kappa(F)^\lambda} \tag{2.1}$$

where $\kappa(F)$ and λ are constants determined from the A-group structure. Furthermore

if $\exp \mu = \sum_{n=0}^{\infty} \frac{\mu^n}{n!}$ we have

$$\|\exp \mu\| \leq \exp(\kappa(F)^\lambda r(\mu)) \tag{2.2}$$

PROOF. First we observe that for any $n \in N$

$$\begin{aligned} \|\mu^n\| &= \sum_{x \in G} |\mu^n(x)| \\ &\leq \#(F^n)^{1/2} \|\mu^n\|_2 \\ &\leq \#(F^n)^{1/2} \sup \|\mu^n * g\|_2, (g \in l_2(G), \|g\|_2 = 1) \\ &\leq \#(F^n)^{1/2} \rho(\mu)^n \end{aligned}$$

where $\|\cdot\|_2$ is the norm in $l_2(G)$ and $\rho(\mu)$ is the norm of the left regular representation, i.e. the norm of the operator $\mu: l_2(G) \rightarrow l_2(G): f \rightarrow \mu * f$. Since $\rho(\mu) \leq r(\mu)$ we have

$$\|\mu^n\| \leq \#(F^n)^{1/2} r(\mu)^n. \tag{2.3}$$

Now by (2.3)

$$\begin{aligned} \left| \left| \delta + \mu + \mu^2 + \dots \right| \right| &< 1 + \#(\mathbb{F})^{1/2} r(\mu) + \\ &+ \#(\mathbb{F}^2)^{1/2} r(\mu)^2 + \dots \\ &< \sum_{n=0}^{\infty} \binom{\kappa(\mathbb{F})+n-1}{n}^\lambda r(\mu)^n \end{aligned}$$

Now let $K = \kappa(\mathbb{F})$ we see that $\binom{K+n-1}{n}^\lambda < \binom{K^\lambda+n-1}{n}$. In fact,

$$\begin{aligned} \binom{K+n-1}{n}^\lambda &= \prod_{j=1}^n \left(\frac{K-1}{j} + 1 \right)^\lambda \\ &= \prod_{j=1}^n \left(1 + \sum_{m=1}^{\lambda} \binom{\lambda}{m} \left(\frac{K-1}{j} \right)^m \right) \\ &< \prod_{j=1}^n \left(\frac{1}{j} [(K-1)+1]^\lambda + \frac{j-1}{j} \right) \\ &< \prod_{j=1}^n \left(\frac{K^\lambda-1}{j} + 1 \right) \\ &< \binom{K^\lambda+n-1}{n} \end{aligned}$$

Thus

$$\left| \left| \delta + \mu + \mu^2 + \dots \right| \right| < \sum_{n=0}^{\infty} \binom{\kappa(\mathbb{F})^\lambda+n-1}{n} r(\mu)^n$$

and since $r(\mu) < 1$, by the binomial formula, $(\delta - \mu)^{-1}$ exists and we obtain (2.1).

To see (2.2) we observe that since $\kappa > 1$ and $j > 1$, $\left(\frac{\kappa-1}{j} + 1 \right) < \kappa$ and

$$\begin{aligned} \binom{\kappa(\mathbb{F})+n-1}{n}^\lambda &< \prod_{j=1}^n \left(\frac{\kappa(\mathbb{F})-1}{j} + 1 \right)^\lambda \\ &< \kappa(\mathbb{F})^{n\lambda} \end{aligned} \tag{2.4}$$

Thus by (2.3) and (2.4)

$$\begin{aligned} \left| \left| \exp \mu \right| \right| &< \sum_{n=0}^{\infty} \frac{\#(\mathbb{F}^n)}{n!} r(\mu)^n \\ &< \sum_{n=0}^{\infty} \frac{1}{n!} \binom{\kappa(\mathbb{F})+n-1}{n}^\lambda r(\mu)^n \\ &< \sum_{n=0}^{\infty} \frac{1}{n!} (\kappa(\mathbb{F})^\lambda r(\mu))^n \\ &= \exp(\kappa(\mathbb{F})^\lambda r(\mu)). \end{aligned}$$

3. THE CLASS OF GROUPS A.

In this section, we show that [FC] groups and nilpotent groups are A-groups. We also show that the class A is closed under finite extension.

First we examine the growth of [FC] groups.

PROPOSITION 3.1. Let G be an [FC] group and F be a finite subset of G . Then

$$\#(F^r) < \binom{k+r-1}{r} \quad (r \in \mathbb{N})$$

where $k = \# \left(\bigcup_{f \in F} [f] \right)$, $[f]$ is the conjugacy class of f .

PROOF. We show that if $[F] = \bigcup_{f \in F} [f]$

$$\#(F^2) < \#([F]^2) < \binom{k+1}{2}$$

Given $f, g \in F$, $f \neq g$, there is an element g_1 (say) in $[g] \subset [F]$ such that $fgf^{-1} = g_1$, and so $fg = g_1f$.

Hence any 2-word in the elements of $[F]$ consisting of two different letters is equal to another 2-word in the elements of $[F]$.

Thus $[F]^2$ has no more than k elements f^2 and $k(k-1)/2$ elements fg where $f, g \in F$, $f \neq g$. It is clear that

$$\#(F^2) < k + \frac{k(k-1)}{2} = \binom{k+1}{2}.$$

We suppose that the Theorem is true for any $r < n$, we show that it is also true for $r = n$.

We denote by $\phi(g)$ the number of all appearances of a $g \in [F]$ in the words of $([F]^n)'$.

If all elements of $[F]$ had the same chance to appear in $([F]^n)'$ then for each $g \in [F]$, $\phi(g) = \frac{n}{k} ([F]^n)$.

Thus we may consider a $g \in [F]$ such that

$$\#([F]^n) < \frac{k}{n} \phi(g) \tag{3.1}$$

Since for any $f \in [F]$ $f \neq g$, there is some f_1 (say) such that $fg = gf_1$. Hence without loss of generality we may assume that in any word of $([F]^n)'$, either there is no g or all g 's keep the left place of the word. Now from each word of $([F]^n)'$, where g appears, cancel one g . The resulting $(n-1)$ -words form a subset of distinct elements of $[F]^{n-1}$; we denote this set by $g^{-1}([F]^n)'$. Hence from our hypothesis it is clear that

$$\phi(g) < \binom{k+n-2}{n-1} + \phi_1(g) \tag{3.2}$$

where $\phi_1(g)$ is the number of all appearances of g in $g^{-1}([F]^n)'$. Suppose that

$$\phi_1(g) < \frac{n-1}{k} \binom{k+n-2}{n-1} \tag{3.3}$$

then by (3.1) and (3.2) we obtain

$$\#([F]^n) < \frac{k}{n} \left(1 + \frac{n-1}{k} \right) \binom{k+n-2}{n-1} < \binom{k+n-1}{n}$$

and so in this case there is nothing to show.

If the inequality (3.3) does not occur, from (3.1) and (3.2) we have

$$([F]^n) < \frac{k}{n} \left(\frac{k}{n-1} + 1 \right) \phi_1(g) < \frac{k(k+n-1)}{n(n-1)} \phi_1(g)$$

In a similar way we define $\phi_i(g)$ ($1 < i < n-1$) i.e. the number of appearances of g in the collection $g^{-1}([F]^n)$.

As in (3.2)

$$\phi_{i-1}(g) < \binom{k+n-i-1}{n-i} + \phi_i(g) \quad (i=2,3,\dots,n-1)$$

and if for some $i < n-1$

$$\phi_i(g) < \frac{n-i}{k} \binom{k+n-i-1}{n-i} \tag{3.4}$$

it is nothing to show. If the inequality (3.4) does not occur for any $i < n-1$ we observe that

$$\phi_{n-1}(g) < \frac{1}{k} \binom{k}{1}$$

In this case we write

$$\# ([F]^n) < \frac{k(k+n-1) \dots (k+1)}{n \cdot (n-1) \dots 1} = \frac{(k+n-1)!}{(k-1)!n!}$$

and this completes the proof.

COROLLARY 3.1. If G is abelian and F is a finite subset of G then,

$$(\mathbb{F}^r) < \binom{\#F+r-1}{r}, \quad r \in \mathbb{N}$$

PROOF. Clear

LEMMA 3.1. Let G be a discrete group with a normal subgroup K such that G/K is abelian, and let π be the canonical map $\pi:G \rightarrow G/K$. If $F \subset G$ is such that $\#F = \# \pi(F)$, then the number of all r -words in the elements of F ($r \in \mathbb{N}$), in a given class of G modulo K can not be greater than

$$\binom{\#F+r-1}{r}.$$

PROOF. Let $\tilde{x} \in G/K$ fixed and $J_r = \pi^{-1}(\tilde{x}) \cap F^r$

Note that J_r and F^r in this proof mean collections of r -words, which as elements of G may not be distinct.

We shall denote by $\# J_r$ the cardinality of J_r , and we shall show that

$$\# ([F]^n) < \binom{\#F+1}{2} \tag{3.5}$$

Let $x,y \in F$ and $xy \in J_2$, then $yx \in J_2$; in fact since G/K is abelian $\pi(xy) = \pi(x)\pi(y) = \pi(yx)$. Now suppose that there is a $z \in F$ such that x,z (or z,x) is in J_2 , i.e. $\pi(x,y) = \pi(x,z)$ and so $\pi(y) = \pi(z)$ and $\# \pi(F) < \# F$ -contradiction. Hence it is clear that $\# J_2 < \# F$ and (3.5) follows.

We suppose that Lemma 3.1 is true for any $r > n-1$ and we show this for $r=n$. For

for some $\tilde{g} \in G/K$, let

$$J_n = \pi^{-1}(\tilde{g}) \cap F^n.$$

If all elements of F had the same chance to appear in J_n , then the number of all appearances $\phi(x)$, $x \in F$, of x in the words of J_n should be

$$\phi(x) = \frac{n}{\#F} \#(J_n)$$

We consider a $x \in F$ such that

$$\#J_n < \frac{\#F}{n} \phi(x) \tag{3.6}$$

From the set of all words in J_n where x has at least one entry we cancel one x . We denote by ${}_xJ_n$ the collection of all the resulting $(n-1)$ -words. We show that ${}_xJ_n$ is in a class of G modulo K .

Let w_1, w_2, w'_1, w'_2 , be words in the elements of F such that $w_1 \times w_2, w'_1 \times w'_2 \in J_n$. Since $\pi(w_1 \times w_2) = \pi(w'_1 \times w'_2)$, we have $\pi(w_1 w_2) = \pi(w'_1 w'_2)$ and ${}_xJ_n$ is as we claimed.

Now, the set ${}_xJ_n$ by our inductive hypothesis has cardinality no greater than $\binom{\#F+n-2}{n-1}$ and so

$$\phi(x) < \binom{\#F+n-2}{n-1} + \phi_1(x)$$

where $\phi_1(x)$ is the number of appearances of x in ${}_xJ_n$.

As in Proposition (3.1), (3.7) in the case where

$$\phi_1(x) < \frac{n-1}{\#F} \binom{\#F+n-2}{n-1}$$

it is nothing to show. If the inequality above is not true by (3.6) and (3.7) we obtain

$$\#J_n < \frac{\#F}{n} \frac{\#F+n-1}{n-1} \phi_1(x)$$

We complete the proof in the same arguments as in Proposition 3.1.

PROPOSITION 3.2. Any nilpotent group G is an A -group with $\kappa(F) = \#F$ and

$\lambda = 2q-1$, where q is the index of G and F is a finite set.

PROOF. Let $G = A_0 \supset A_1 \supset \dots \supset A_{q-1} \supset A_q = \{e\}$ be the normal series of a nilpotent group G of index q . We note that A_{i-1}/A_i is the center of G/A_i ($1 < i < q-1$) and we denote by π_i the canonical map $G \rightarrow G/A_i$.

It is obvious that for any $F \subset G$ and $r \in \mathbb{N}$

$$\#(F^r) < \#(\pi_1(F^r)) \max \{ \# \pi^{-1}(\tilde{g}) \cap F^r : \tilde{g} \in G/A_1 \}. \tag{3.8}$$

We denote by $|F^r|_1$ the RHS of (3.8).

We shall show that there are q positive integers m_1, m_2, \dots, m_q such that

$\#(F) = m_1 + m_2 + \dots + m_q$ and

$$|F^r|_1 \leq \binom{m_1+r-1}{r} \binom{m_2+r-1}{r} \dots \binom{m_{q-1}+r-1}{r} \binom{m_q+r-1}{r} \tag{3.9}$$

If $q=1$, i.e. G is abelian. Corollary (3.1) implies (3.9). We suppose that (3.9) is true for each nilpotent group of index $q = p - 1$.

Let G be nilpotent of index p . Since G/A_1 is abelian by Lemma 3.1 if $\#F < \# \pi_1(F)$.

We have,
$$|F^r|_1 \leq \binom{\#F+r-1}{r}^2.$$

Let $\#(\pi_1(F)) = m_1 < \#(F)$, then F can be written as

$$F = \{x_i \alpha_j : 1 \leq i \leq m_1, 1 \leq j \leq \#(F)-m_1\}$$

where $\pi_1(x_i) \neq \pi_1(x_j)$ $i \neq j$, $1 \leq j \leq m_1$, and all α_j 's are in A_1 . By (3.8) we obtain,

$$|F^r|_1 \leq \binom{m_1+r-1}{r} \# J_1 \tag{3.10}$$

where $J_1 = {}^{-1}_1(g) F^r$, for some $g \in G/A_1$. Any element of J_1 can be written as

$$x_{i_1} \alpha_{j_1} x_{i_2} \alpha_{j_2} \dots x_{i_r} \alpha_{j_r} \tag{3.11}$$

where $(x_{i_1} x_{i_2} \dots x_{i_r}) \in \tilde{g}$, each i_t ($1 \leq t \leq r$) is one of $1, 2, \dots, m_1$ and each j_t is one of $1, 2, \dots, \#F-m_1$.

By Lemma (3.1) the cardinality of all $x_{i_1} \dots x_{i_r}$ in \tilde{g} is $< \binom{m_1+r-1}{r}$ and so

$$\# J'_1 \leq \binom{m_1+r-1}{r} \# J'_1$$

where

$$J'_1 = x_{i_1} F_1 x_{i_2} F_1 \dots x_{i_r} F_1$$

$x_{i_1} x_{i_2} \dots x_{i_r}$ is fixed suitably chosen from (3.11) and belongs to \tilde{g} ;

$$F_1 = \{\alpha_j : 1 \leq j \leq \#F-m_1\}$$

Note that if $q = 2$, then A_1 is the center of G , all α_j 's commute with x_i 's and by Corollary (3.1), $|F^r|_1 < \binom{\#F-m_1+r-1}{r}$; thus by (3.10) and (3.12), (3.9) follows.

As in (3.8), for some $\tilde{g} \in G/A_2$ we have

$$\#J'_1 \leq \# \pi_2(J'_1) \# \{\pi_2^{-1}(\tilde{g}) \cap J'_1\} \tag{3.13}$$

Since A_1/A_2 is the center of G/A_2 and $F_1 \subset A_1$, by (3.10), (3.12) and (3.13) we obtain

$$|F^r|_1 \leq \binom{m_1+r-1}{r}^2 \cdot |F^r|_1 \tag{3.14}$$

where $|F_1^r|_1$ is defined as $|F^r|_1$.

Since A_1 is a nilpotent of index $p-1$ we apply our inductive hypothesis in (3.14) and for $q = p$ we obtain (3.9).

Now if we replace m_1, \dots, m_q in (3.9) by $\#F$ we see that G is an A group with constants $k = 1$ and $\lambda = 2q-1$.

PROPOSITION 3.3. The class of A -groups is closed under extensions by finite groups.

PROOF. We may write $G/A = \{d_1A, d_2A, \dots, d_sA\}$ where d_1, d_2, \dots, d_s are s representatives of all the different classes of G/A ; without loss of generality let $d_s = e$ the unit of G . We may also write

$$d_i d_j = \alpha(i, j) d(i, j) \quad (1 < i, j < s)$$

where each $d(i, j)$ is one of d_1, \dots, d_s and each $\alpha(i, j)$ is in A .

Let $F = \{d_{i_1} x_1, \dots, d_{i_m} x_m\}$, $\#F = m$, each d_{i_t} is one of d_1, \dots, d_s and $x_t \in A$ ($1 < t < m$).

Let $\langle x_i \rangle = \{d_j^{-1} x_i d_j : j = 1, 2, \dots, s\}$ ($1 < i < m$) and

$$d_{j_1} x_{i_1} d_{j_2} x_{i_2} \dots d_{j_r} x_{i_r} \quad (3.15)$$

be a typical r -word in the elements of F . Each word in (3.15) is in the set

$\langle x_{i_1} \rangle \alpha(j_1, j_2) d(j_1, j_2) x_{i_2} \dots d_{j_r} x_{i_r}$ or in

$\langle x_{i_1} \rangle \alpha(j_1, j_2) \langle x_{i_2} \rangle \alpha(\cdot, j_3) \dots d_{j_r} x_{i_r}$ or finally in

$$\langle x_{i_1} \rangle \langle x_{i_2} \rangle \dots \langle x_{i_r} \rangle \text{ a.d.} \quad (3.16)$$

where $\alpha \in A$ and $d \in \{d_1, d_2, \dots, d_s\}$.

It is clear that the cardinality of the r -words in the elements of F , as in (3.15) is less than the cardinality of the words in (3.16) in the elements of $\{\langle x_1 \rangle, \dots, \langle x_m \rangle\}$, which is a subset of the A -group, A . Thus G inherits the growth of A .

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