

SOME BAZILEVIĆ FUNCTIONS OF ORDER BETA

S.A. HALIM

Department of Mathematics and Computer Sciences
University of Wales
University College of Swansea
Singleton Park Swansea SA2, 8PP, U.K.

(Paper Received April 4, 1988 and in Revised form September 13, 1988)

ABSTRACT. Distortion theorems and coefficient estimates are obtained for a new class of Bazilevič functions.

KEYWORDS AND PHRASES. Bazilevič Functions, Functions Whose Derivative has Positive Real Part, Coefficient Estimates.

1980 AMS Classification Code. Primary 30C45.

1. INTRODUCTION.

Let S be the class of normalized functions regular and univalent in the unit disc $D = \{z : |z| < 1\}$ and S^* the subclass of starlike functions. Denote by $P(\beta)$, the class of functions which are regular in D and such that for $h \in P(\beta)$, $h(0) = 1$ and $\operatorname{Re} h(z) > \beta$ for $z \in D$. We write $P = P(0)$.

Bazilevič [1] showed that the class of normalized regular functions f with representation

$$f(z) = \left(\alpha \int_0^z p(t) g(t)^\alpha t^{-1} dt \right)^{\frac{1}{\alpha}} \quad (1.1)$$

when $\alpha > 0$, $g \in S^*$ and $p \in P$ for $z \in D$ forms a subclass of S . We denote this class of functions by $B(\alpha)$. See also [2].

Let $\alpha > 0$. Then it follows easily from (1.1) that $f \in B(\alpha)$ if, and only if, there exists $g \in S^*$ such that for $z \in D$

$$\operatorname{Re} \frac{z f'(z)}{f(z)^{1-\alpha} g(z)^\alpha} > 0. \quad (1.2)$$

In [3], Singh considered the subclass $B_1(\alpha)$ of $B(\alpha)$ obtained by taking $g(z) \equiv z$ in (1.2). Thus $f \in B_1(\alpha)$ if, and only if, for $\alpha > 0$ and $z \in D$

$$\operatorname{Re} \frac{z^{1-\alpha} f'(z)}{f(z)^{1-\alpha}} > 0.$$

We extend this class of functions as follows:

DEFINITION. Let f be regular in D with

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.3)$$

Then if $\alpha > 0$ and $0 < \beta < 1$, $f \in B_1(\alpha, \beta)$ if, and only if, for $z \in D$

$$\operatorname{Re} \frac{z^{1-\alpha} f'(z)}{f(z)^{1-\alpha}} > \beta. \quad (1.4)$$

We note that $B_1(1, 0) = R$, the class of functions whose derivative has real part [4]. $B_1(1, \beta)$ was considered in [5]. Zamorski [6] and Thomas [7] solved the coefficient problem for $f \in B(\frac{1}{N})$, in the case when N is a positive integer. In [7], sharp distortion theorems were obtained for $f \in B_1(\alpha)$ for $\alpha > 0$. The object of this paper is to extend these results to the class $B_1(\alpha, \beta)$. The class $B_1(\alpha, \beta)$ has also recently been considered in [8].

2. RESULTS.

Distortion Theorems

THEOREM 1. Let $f \in B_1(\alpha, \beta)$. Then for $z = re^{i\theta} \in D$, $0 < r < 1$,

$$(i) \quad Q_2(r)^\alpha < |f(z)| < Q_1(r)^\alpha,$$

$$(ii) \quad \text{if } 0 < \alpha < 1,$$

$$r^{\alpha-1} Q_2(r)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1-r)(1-\beta)}{(1+r)} + \beta \right) < |f'(z)| < r^{\alpha-1} Q_1(r)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1+r)(1-\beta)}{(1-r)} + \beta \right)$$

and if $\alpha > 1$

$$r^{\alpha-1} Q_1(r)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1-r)(1-\beta)}{(1+r)} + \beta \right) < |f'(z)| < r^{\alpha-1} Q_2(r)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1+r)(1-\beta)}{(1-r)} + \beta \right)$$

where

$$Q_1(r) = \alpha \int_0^r \rho^{\alpha-1} \left(\frac{(1+\rho)(1-\beta)}{(1-\rho)} + \beta \right) d\rho,$$

and

$$Q_2(r) = \alpha \int_0^r \rho^{\alpha-1} \left(\frac{(1-\rho)(1-\beta)}{(1+\rho)} + \beta \right) d\rho.$$

Equality holds in all cases for the function f_ϕ , defined by

$$f_\phi(z) = \left(\alpha \int_0^z t^{\alpha-1} \left(\frac{(1+te^{i\phi})(1-\beta)}{(1-te^{i\phi})} + \beta \right) dt \right)^{\frac{1}{\alpha}} \quad (2.1)$$

where $\phi = 0$ or π .

PROOF.

(i) Since $f \in B_1(\alpha, \beta)$, and it follows from (1.4) that

$$(1-\beta)p(z) = \frac{z^{1-\alpha} f'(z)}{f(z)^{1-\alpha}} - \beta$$

for $z \in D$ and $p \in P$.

Thus

$$f(z)^\alpha = \alpha \int_0^z t^{\alpha-1} (p(t)(1-\beta) + \beta) dt \quad (2.2)$$

and since $|p(z)| < \frac{1+r}{1-r}$ for $z \in D$, (see eg. [9]),

$$\begin{aligned} |f(z)|^\alpha &< \alpha \int_0^r \rho^{\alpha-1} \left(\frac{(1+\rho)(1-\beta)}{1-\rho} + \beta \right) d\rho \\ &= Q_1(r). \end{aligned}$$

To obtain the left-hand inequality in (i), write

$$h(z) = \frac{z^{1-\alpha} f'(z)}{f(z)^{1-\alpha}}. \quad (2.3)$$

Then (1.4) shows that $h \in p(\beta)$. Thus from, [5] (Theorem 1 with $c=1-2\beta$ and $n=1$), we obtain

$$\frac{(1-r)(1-\beta)}{(1+r)} + \beta < |h(z)| < \frac{(1+r)(1-\beta)}{(1-r)} + \beta. \quad (2.4)$$

Hence from (2.3) and (2.4) we have

$$\left| \frac{d}{dz} [f(z)]^\alpha \right| > \alpha r^{\alpha-1} \left(\frac{(1-r)(1-\beta)}{(1+r)} + \beta \right). \quad (2.5)$$

Now let z_1 , $|z_1| = r$ be chosen so that $|f(z_1)^\alpha| < |f(z)^\alpha|$ for all z with $|z| = r$. Writing $\omega = f(z_1)^\alpha$, it follows that since f is univalent, the line segment λ from 0 to ω lies entirely in the image of D . Let \mathfrak{z} be the pre-image of λ , then by (2.5)

$$\begin{aligned}
 |f(z)|^\alpha &> |f(z_1)|^\alpha = \int_\lambda |d\omega| = \int_L \left| \frac{d\omega}{dz_1} \right| |dz_1| \\
 &> \int_0^r \alpha \rho^{\alpha-1} \left(\frac{(1-\rho)(1-\beta)}{(1+\rho)} + \beta \right) d\rho
 \end{aligned}$$

which is the left-hand inequality in (i).

(ii) From (2.3) we have for $z = re^{i\theta}$

$$|f'(z)| = r^{\alpha-1} |f(z)|^{1-\alpha} |h(z)| \tag{2.6}$$

if $0 < \alpha < 1$, the inequalities follow at once from (2.6), (2.4) and (i). If $\alpha > 1$, (i) gives

$$Q_1(r)^{\frac{1-\alpha}{\alpha}} < |f(z)|^{1-\alpha} < Q_2(r)^{\frac{1-\alpha}{\alpha}} . \tag{2.7}$$

Applying (2.4) and (2.7) to (2.6) gives the required result. Equality is attained in and (i) for f_0 and in (ii) for f_0 when $0 < \alpha < 1$ and for f_π when $\alpha > 1$.

The following shows that as $\alpha \rightarrow 0$ the bounds in Theorem 1 are asymptotic to the distortion theorems for starlike functions of order $\beta > 0$ (see eg. [9]).

THEOREM 2. For $0 < r < 1$, let $Q_1(r)$ and $Q_2(r)$ be defined as in Theorem 1. Then as $\alpha \rightarrow 0$

$$\begin{aligned}
 \text{(i)} \quad Q_1(r)^{\frac{1}{\alpha}} &\sim \frac{r}{(1-r)^{2(1-\beta)}} , \\
 \text{(ii)} \quad Q_2(r)^{\frac{1}{\alpha}} &\sim \frac{r}{(1+r)^{2(1-\beta)}} , \\
 \text{(iii)} \quad Q_1(r) &\sim Q_2(r) \sim 1.
 \end{aligned}$$

PROOF.

We prove (i), since (ii) and (iii) are similar.

As $\alpha \rightarrow 0$,

$$\begin{aligned}
 Q_1(r)^{\frac{1}{\alpha}} &= \alpha \int_0^r \rho^{\alpha-1} \left(\frac{(1+\rho)(1-\beta)}{1-\rho} + \beta \right) d\rho \\
 &= r \left(1 + 2\alpha(1-\beta)r^{-\alpha} \int_0^r \frac{\rho^\alpha}{1-\rho} d\rho \right)^{\frac{1}{\alpha}} \\
 &\sim r(1-2\alpha(1-\beta)r^{-\alpha} \log(1-r))^{\frac{1}{\alpha}} \\
 &\sim re^{-2(1-\beta)\log(1-r)} = \frac{r}{(1-r)^{2(1-\beta)}} .
 \end{aligned}$$

COROLLARY.

Suppose that $f(z) \neq \omega$ for $z \in D$, then

$$|\omega| > Q_2(1)^{\frac{1}{\alpha}} \sim 4^{\beta-1} \text{ as } \alpha \rightarrow 0.$$

PROOF.

Let $\alpha > 0$, and ω be a point on the boundary of $f(D)$ closest to the origin. Let L_1 denote the straight line from 0 to ω and L its pre-image in D . Then $|\omega| > |F(z)|$ for $z \in L \cap D$. Since the circle $|z| = r$ intersects L , at least

once, Theorem 1 (i) gives $|\omega| > Q_2(r)^{\frac{1}{\alpha}}$.

Thus Theorem 2 (ii) gives

$$|\omega| > Q_2(1)^{\frac{1}{\alpha}} \sim 4^{\beta-1} \text{ as } \alpha \rightarrow 0.$$

3. A COEFFICIENT THEOREM

Notation: $\sum_{n=0}^{\infty} \alpha_n z^n \ll \sum_{n=0}^{\infty} \beta_n z^n$ means $|\alpha_n| < |\beta_n|$ for $n > 0$.

THEOREM 3. Let $f \in B_1(\frac{1}{N}, \beta)$ and be given by (1.3) where N is a positive integer. Suppose also that for $z \in D$,

$$f_0(z) = z + \sum_{n=0}^{\infty} \gamma_n z^n \text{ where } f_0(z) \text{ is given by (2.1).}$$

Then (i) $f(z) \ll f_0(z)$,

and (ii) $\gamma_n \sim (\frac{2(1-\beta)}{N})^N (\frac{N}{n}) (\log n)^{N-1}$ as $n \rightarrow \infty$.

PROOF.

(i) Thomas [7], proved that if $|\alpha_n| < |\beta_n|$, then for $m = 1, 2, 3, \dots$,

$$(\sum_{n=1}^{\infty} \alpha_n z^n)^m \ll (\sum_{n=1}^{\infty} \beta_n z^n)^m.$$

Write $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$. Then (2.2) gives

$$\begin{aligned} f(z)^{\frac{1}{N}} &= \frac{1}{N} \int_0^z t^{\alpha-1} [(1 + \sum_{k=1}^{\infty} p_k t^k)^{(1+\beta) + \beta}] dt \\ &= \frac{1}{N} [N(1-\beta)z^{\frac{1}{N}} + (1-\beta) \sum_{k=1}^{\infty} (\frac{p_k z^{k+\frac{1}{N}}}{k + \frac{1}{N}}) + \beta N z^{\frac{1}{N}}] \\ &= z^{\frac{1}{N}} (1 + \frac{(1-\beta)}{N} \sum_{k=1}^{\infty} \frac{p_k z^k}{(k+\frac{1}{N})}). \end{aligned}$$

Thus

$$f(z) = z (1 + \frac{(1-\beta)}{N} \sum_{k=1}^{\infty} \frac{p_k z^k}{(k+\frac{1}{N})})^N$$

and since $p \in P$, we have $|P_k| < 2$ [6]. Hence

$$f(z) = z \left(1 + \frac{(1-\beta)}{N} \sum_{k=1}^{\infty} \frac{P_k z^k}{(k+\frac{1}{N})} \right)^N \left(z \left(1 + \frac{(1-\beta)}{N} \sum_{k=1}^{\infty} \frac{2z^k}{(k+\frac{1}{N})} \right)^N = f_0(z).$$

(ii) Putting $\alpha = \frac{1}{N}$ in (2.1), we have

$$\begin{aligned} f_0(z) &= z + \sum_{n=2}^{\infty} \gamma_n z^n = z \left(1 + \frac{2(1-\beta)}{N} \sum_{n=1}^{\infty} \frac{z^n}{(n+\frac{1}{N})} \right)^N \\ &= z \sum_{\nu=0}^{\infty} \binom{N}{\nu} \left(\frac{2(1-\beta)}{N} \right)^{\nu} \left(\sum_{n=1}^{\infty} \frac{z^n}{(n+\frac{1}{N})} \right)^{\nu}. \end{aligned}$$

Let

$$\left(\sum_{n=1}^{\infty} \frac{z^n}{(n+\frac{1}{N})} \right)^{\nu} = \sum_{n=\nu}^{\infty} D_n(\nu) z^n \quad (\nu = 0, 1, 2, 3, \dots).$$

Thomas [7] proved that $D_n(\nu) \sim \frac{\nu}{N} (\log n)^{\nu-1}$ as $n \rightarrow \infty$ and so this gives

$$\begin{aligned} \gamma_n &= \sum_{\nu=0}^{\infty} \binom{N}{\nu} \left(\frac{2(1-\beta)}{N} \right)^{\nu} D_n(\nu) \\ &\sim \left(\frac{2(1-\beta)}{N} \right)^N \binom{N}{n} (\log n)^{N-1} \text{ as } n \rightarrow \infty. \end{aligned}$$

REFERENCES

1. BAZILEVIČ, I.E., On a case of integrability in quadratures of the Loewner-Kufarev equation, Mat. Sb. 37(79)(1955), 471-476.
2. THOMAS, D.K., On Bazilevič functions, Math. Z. 109(1969), 344-348.
3. SINGH, R., On Bazilevič functions, Proc. Amer. Mat. Soc. 38(1973), 261-271.
4. MACGREGOR, T.H., Functions whose derivative has positive real part, Trans. Amer. Math. Soc. 104 (1962), 532-537.
5. TONTI, NORMAN E. & TRAHAN, DONALD H., Analytic functions whose real parts are bounded below, Math. Z. 115(1970), 252-258.
6. ZAMORSKI, J., On Bazilevič Schlicht functions, Ann. Polon. Math. 12 (1962), 83-90.
7. THOMAS, D.K., On a subclass of Bazilevič functions, Internat. J. Math. and Math. Sci., Vol.8, No. 4 (1985), 799-783.
8. HALIM, ABDUL S., On the coefficients of some Bazilevič functions of order β . To appear.
9. GOODMAN, A.W., Univalent functions, Vol.I., Mariner Publishing Co., Tampa Florida, 1983.
10. POMMERENKE, Ch., Univalent functions, Vandenhoeck and Ruprecht, Göttingen, 1975.

Special Issue on Singular Boundary Value Problems for Ordinary Differential Equations

Call for Papers

The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal's Author Guidelines, which are located at <http://www.hindawi.com/journals/bvp/guidelines.html>. Authors should follow the Boundary Value Problems manuscript format described at the journal site <http://www.hindawi.com/journals/bvp/>. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	May 1, 2009
First Round of Reviews	August 1, 2009
Publication Date	November 1, 2009

Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de

Compostela, Santiago de Compostela 15782, Spain;
juanjose.nieto.roig@usc.es

Guest Editor

Donal O'Regan, Department of Mathematics, National University of Ireland, Galway, Ireland;
donal.oregan@nuigalway.ie