

## FUZZY $\Theta$ -CLOSURE OPERATOR ON FUZZY TOPOLOGICAL SPACES

M.N. MUKHERJEE  
and  
S.P. SINHA

Department of Pure Mathematics  
University of Calcutta  
35, Ballygunge Circular Road  
Calcutta - 700 019, INDIA

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**ABSTRACT.** The paper contains a study of fuzzy  $\Theta$ -closure operator,  $\Theta$ -closures of fuzzy sets in a fuzzy topological space are characterized and some of their properties along with their relation with fuzzy  $\delta$ -closures are investigated. As applications of these concepts, certain functions as well as some spaces satisfying certain fuzzy separation axioms are characterized in terms of fuzzy  $\Theta$ -closures and  $\delta$ -closures.

**KEY WORDS AND PHRASES.** Fuzzy  $\Theta$ -cluster point, fuzzy  $\Theta$ -closure, fuzzy  $\delta$ -closure, q-coincidence, q-neighbourhood.

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### 1. INTRODUCTION.

It is well-known that the concepts of  $\Theta$ -closure and  $\delta$ -closure are useful tools in standard topology in the study of H-closed spaces, Katetov's and H-closed extensions, generalizations of Stone-Weierstrass' theorem etc. For basic results and some applications of  $\Theta$ -closure and  $\delta$ -closure operators we refer to Veličko [1], Dickman and Porter [2], Espelie and Joseph [3] and Sivaraj [4]. Due to varied applicabilities of these operators in formulating various important set-topological concepts, it is natural to try for their extensions to fuzzy topological spaces. With this motivation in mind the concept of  $\Theta$ -closure operator in a fuzzy topological space (due to Chang [5]) was introduced by us in [6] in the light of the notions of quasi-coincidence and q-neighbourhoods of Pu and Liu [7,8]. In the present paper our aim is to continue the same study which ultimately shows that different fuzzy topological concepts can effectively be characterized in terms of fuzzy  $\Theta$ -closure and  $\delta$ -closure operators.

In Section 2 of this paper we develop the concept of fuzzy  $\Theta$ -closure operators and characterize fuzzy  $\Theta$ -closures of fuzzy sets in a fuzzy topological space in different ways. In literature there can be found several definitions of  $T_2$ -spaces in fuzzy setting. We take the definition of fuzzy  $T_2$ -space as given by Ganguly and Saha [9] and become able to successfully characterize it in our context. Fuzzy regularity has been introduced by many workers from different view points, including one by us in [6]. Since our fuzzy regularity along with the fuzzy  $T_1$ -axiom (of [9]) does not yield the above fuzzy  $T_2$ -axiom, we propose to call it "strong  $T_2$ " in fuzzy setting. Fuzzy semiregularity and almost

regularity were also defined in [6]. We characterize fuzzy regularity and these weaker forms of fuzzy regularity in terms of fuzzy  $\Theta$ -closure and  $\delta$ -closure. All these characterizations are incorporated in Section 3 of the paper. Fuzzy weakly continuous functions were first introduced by Azad [10] and were further investigated in [11], whereas the concept of fuzzy  $\Theta$ -continuous functions was initiated in [6]. Section 3 also includes the characterizations of these functions with the help of the notion of fuzzy  $\Theta$ -closures.

We now recall some definitions and results of a fuzzy topological space (henceforth fts, for short)  $(X, T)$  to be used in this paper excepting very standard ones for which we refer to Zadeh [12], Chang [5] and Pu and Liu [7,8]. The interior and closure of a fuzzy set  $A$  in an fts  $(X, T)$  will be denoted by  $\text{Int } A$  and  $\text{Cl } A$  respectively. A fuzzy point [7] with a singleton support  $x$  (say) and value  $\alpha$  ( $0 < \alpha \leq 1$ ) at  $x$  is denoted by  $x_\alpha$ . For a fuzzy set  $A$ , the support and complement of  $A$  are denoted by  $A_o$  and  $A'$  (or  $1 - A$ ) respectively. For a fuzzy point  $x_\alpha$  and a fuzzy set  $A$ , we write  $x_\alpha \in A$  iff  $\alpha \leq A(x)$ , and  $x_\alpha$  is said to be quasi-coincident (q-coincident, for short) with  $A$ , denoted by  $x_\alpha qA$ , iff  $\alpha > A(x)$ .  $A$  is said to be a q-neighbourhood (q-nbd, for short) of  $x_\alpha$  iff there exists a fuzzy open set  $B$  such that  $x_\alpha qB \leq A$ . For two fuzzy sets  $A$  and  $B$ ,  $A \leq B$  iff  $A \not\# B'$ , and a fuzzy point  $x_\alpha \in \text{Cl } A$  iff each q-nbd of  $x_\alpha$  is q-coincident with  $A$  [7]. For the definitions of fuzzy regularly open, regularly closed, semi-open and semi-closed sets we refer to Azad [10]. Simply by  $X$  and  $Y$  we shall mean the fuzzy topological spaces  $(X, T)$  and  $(Y, T_1)$  respectively. The constant fuzzy sets  $0_X$  and  $1_X$  are defined by  $O_X(y) = 0$  and  $1_X(y) = 1$ , for each  $y \in X$ .

## 2. FUZZY $\Theta$ -CLOSURE AND ITS PROPERTIES.

**DEFINITION 2.1.** A fuzzy point  $x_\alpha$  is said to be a fuzzy  $\Theta$ -cluster point ( $\delta$ -cluster point [13]) of a fuzzy set  $A$  iff closures of every open q-nbd (resp. iff every regularly open q-nbd) of  $x_\alpha$  is q-coincident with  $A$ .

The union of all fuzzy  $\Theta$ -cluster ( $\delta$ -cluster) points of  $A$  is called the fuzzy  $\Theta$ -closure of  $A$  and is denoted by  $[A]_\Theta$  (resp.  $[A]_\delta$ ). A fuzzy set  $A$  will be called fuzzy  $\Theta$ -closed ( $\delta$ -closed) iff  $A = [A]_\Theta$  (resp.  $A = [A]_\delta$ ). It is known [6] that for any fuzzy set  $A$  in an fts  $X$ ,  $\text{Cl } A \leq [A]_\delta \leq [A]_\Theta$ , but the reverse implications are false (see [6] and [13]). However, it is true (see [6]) that for a fuzzy open set  $A$  in an fts  $X$ ,  $\text{Cl } A = [A]_\delta = [A]_\Theta$ .

**THEOREM 2.2.** In an fts  $(X, T)$ , the following hold:

- Finite union and arbitrary intersection of  $\Theta$ -closed sets in  $X$  is fuzzy  $\Theta$ -closed.
- For two fuzzy sets  $A$  and  $B$  in  $X$ , if  $A \leq B$  then  $[A]_\Theta \leq [B]_\Theta$ .
- The fuzzy sets  $0_X$  and  $1_X$  are fuzzy  $\Theta$ -closed.

**PROOF.** The straightforward proofs are omitted.

**REMARK 2.3.** The complements of fuzzy  $\Theta$ -closed sets in an fts  $(X, T)$  induce a fuzzy topology  $T_\Theta$  (say) which is coarser than the fuzzy topology  $T$  of the space. Again, for a fuzzy set  $A$  in  $X$ ,  $[A]_\Theta$  is evidently fuzzy closed but not necessarily fuzzy  $\Theta$ -closed as is seen from the next example. Thus, fuzzy  $\Theta$ -closure operator is not a Kuratowski closure operator. However, it will be shown in the next section that for any fuzzy set  $A$  in an fts  $X$ ,  $[A]_\Theta$  is fuzzy  $\Theta$ -closed if the space  $X$  is fuzzy regular (see Corollary 3.6), or iff the space  $X$  is fuzzy almost regular (see Theorem 3.10).

**EXAMPLE 2.4.** Let  $X = \{a, b, c\}$  and  $T = \{0_X, 1_X, A, B\}$ ,

where

$$A(a) = 0.5, A(b) = 0.6, A(c) = 0.2$$

and

$$B(a) = 0.4, B(b) = 0.5, B(c) = 0.1.$$

Let  $U$  be any fuzzy set given by,  $U(a) = U(b) = 0.3$  and  $U(c) = 0$ . Then,  $a_{.6} \in [U]_\Theta, a_{.8} \notin [U]_\Theta$ , but  $a_{.8} \in [a_{.6}]_\Theta \leq [[U]]_\Theta$ . Thus,  $[U]_\Theta \neq [[U]]_\Theta$ . Hence,  $[U]_\Theta$  is not fuzzy  $\Theta$ -closed.

In the following example, we observe a deviation from the corresponding established result [3] in general topology that  $x \in [y]_\Theta$  iff  $y \in [x]_\Theta$ , if  $x, y$  are two points in a topological space.

EXAMPLE 2.5. Let  $X$  be an ordinary set with at least two distinct points  $a, b$ . Consider the fuzzy topology  $T = \{0_X, 1_X A\}$ , where  $A(a) = \frac{1}{2}$ ,  $A(b) = \frac{2}{5}$  and  $A(x) = 0$ , for  $x \neq a, b (x \in X)$ . Let us consider the fuzzy points  $a_{\frac{1}{12}}$  and  $b_{\frac{4}{5}}$ . It can be checked that  $a_{\frac{1}{12}} \in [b_{\frac{4}{5}}]_{\Theta}$ , but  $b_{\frac{4}{5}} \notin [a_{\frac{1}{12}}]_{\Theta}$ ,

THEOREM 2.6. For any fuzzy set  $A$  in an fts  $(X, T)$ ,  $[A]_{\Theta} = \cap \{[U]_{\Theta} : U \in T \text{ and } A \leq U\}$ .

PROOF. Obviously, L.H.S.  $\leq$  R.H.S. Now, if possible let  $x_{\alpha} \in$  R.H.S. but  $x_{\alpha} \notin [A]_{\Theta}$ . Then there exists an open q-nbd  $V$  of  $x_{\alpha}$  such that  $\text{Cl } \not\# A$  and hence  $A \leq 1 - C1V$ . Then  $x_{\alpha} \in [1 - C1V]_{\Theta}$  and consequently,  $\text{Cl } Vq(1 - C1V)$  which is impossible.

According to Pu and Liu [7] a function  $S: D \rightarrow J$  is called a fuzzy net in  $X$ , where  $(D, \geq)$  is a directed set and  $J$  denote the collection of all fuzzy points in  $X$ . It is denoted by  $\{S_n, n \in D\}$  or simply by  $(S, D)$ . We now set the following:

DEFINITION 2.7. Let  $\{S_n, n \in D\}$  be a fuzzy net and  $x_{\alpha}$  a fuzzy point in  $X$ .

- (a)  $x_{\alpha}$  is called a  $\Theta$ -cluster point of the fuzzy net iff for every open q-nbd  $W$  of  $x_{\alpha}$  and for any  $n \in D$ , there exists  $m \geq n$  ( $m \in D$ ) such that  $S_m q C1W$ .
- (b) The fuzzy net is said to be  $\Theta$ -converge to  $x_{\alpha}$  if for any open q-nbd  $U$  of  $x_{\alpha}$ , exists  $m \in D$  such that  $S_n q C1U$ , for all  $n \geq m$  ( $n \in D$ ). This is denoted by  $S_{\Theta}^{\Theta} x_{\alpha}$ .

THEOREM 2.8. A fuzzy point  $x_{\alpha}$  is a  $\Theta$ -cluster point of a fuzzy net  $\{s_n, n \in D\}$  in  $X$  iff there is a subnet of  $\{S_n, n \in D\}$ , which  $\Theta$ -converges to  $x_{\alpha}$ .

PROOF. Let  $x_{\alpha}$  be a  $\Theta$ -cluster point of the given fuzzy net. Let  $Qx_{\alpha}$  denote the set of closures of all open q-nbds of  $x_{\alpha}$ . Now for any member  $A$  of  $Qx_{\alpha}$ , there exists an element  $S_n$  of the net such that  $S_n q A$ . Let  $E$  denote the set of all ordered pairs  $(n, A)$  with the above property, i.e.,  $n \in D, A \in Qx_{\alpha}$  and  $S_n q A$ . Then  $(E, \gg)$  is a directed set, where  $(m, A) \gg (n, B), ((m, A), (n, B) \in E)$  iff  $m \geq n$  in  $D$  and  $A \leq B$ . Then  $T: (E, \gg) \rightarrow (X, T)$  given by  $T(m, A) = S_m$  can be checked by a subnet of  $\{S_n, n \in D\}$ . To show that  $T_{\Theta}^{\Theta} x_{\alpha}$ , let  $V$  be any open q-nbd of  $x_{\alpha}$ . Then there exists  $n \in D$  such that  $(n: C1V) \in E$  and then  $S_n q C1V$ . Now, for any  $(m, A) \gg (n, C1V), T(m, A) = S_m q A \leq C1V$ . Hence,  $T_{\Theta}^{\Theta} x_{\alpha}$ . Converse is clear.

THEOREM 2.9. Let  $A$  be a fuzzy set in  $X$ . A fuzzy point  $x_{\alpha} \in [A]_{\Theta}$  iff there exists a fuzzy net in  $A$ ,  $\Theta$ -converging to  $x_{\alpha}$ .

PROOF. Let  $x_{\alpha} \in [A]_{\Theta}$ . For each open q-nbd  $U$  of  $x_{\alpha}, C1U q A$ . That is, there exist  $y^U \in A_{\Theta}$  and real number  $\beta_U$  with  $0 < \beta_U \leq A(y^U)$  such that  $y_{\beta_U}^U \in A$  and  $y_{\beta_U}^U q C1U$ . We choose and fix one such  $y_{\beta_U}^U$  for each  $U$ . Let  $D$  denote the set of all open q-nbds of  $x_{\alpha}$ . Then  $(D, \geq)$  is directed under inclusion relation, i.e., for  $B, C \in D, B \geq C$  iff  $B \leq C$ . Then  $\{y_{\beta_U}^U \in A: y_{\beta_U}^U q C1U \text{ and } U \in D\}$  is a fuzzy net in  $A$  such that it  $\Theta$ -converges to  $x_{\alpha}$ . Converse is straightforward even if  $x_{\alpha}$  is a  $\Theta$ -cluster point of the fuzzy net in  $A$ .

### 3. CHARACTERIZATIONS OF CERTAIN SEPARATION AXIOMS AND FUNCTIONS IN TERMS OF FUZZY $\Theta$ -CLOSURE AND $\delta$ -CLOSURE.

DEFINITION 3.1. [9] An fts  $X$  is called fuzzy strongly  $T_2$  iff for any two distinct fuzzy points  $x_{\alpha}$  and  $y_{\beta}$  in  $X$ : whenever  $x \neq y, x_{\alpha}$  and  $y_{\beta}$  have fuzzy open nbds  $U$  and  $V$  respectively such that  $U \not\# V$ ; and when  $x = y, \alpha < \beta$  (say), there exist fuzzy open sets  $U$  and  $V$  such that  $x_{\alpha} \in U, y_{\beta} q V$  and  $U \not\# V$ .

LEMMA 3.2. For any two fuzzy open sets  $A$  and  $B$  in an fts  $(X, T)$ ,  $A \not\# B \Rightarrow C1A \not\# B$  and  $A \not\# C1B$ .

THEOREM 3.3. An fts  $(X, T)$  is fuzzy strongly  $T_2$  iff every fuzzy point of  $X$  is fuzzy  $\Theta$ -closed, and for  $x, y \in X$  with  $x \neq (C1U)_{\Theta}$ .

PROOF. Let  $X$  be fuzzy strongly  $T_2$ , and let  $x_{\alpha}$  be a fuzzy point in  $X$ . In order to show that  $[x_{\alpha}]_{\Theta} = x_{\alpha}$ , it suffices to establish that for any fuzzy point  $y_{\beta}, y_{\beta} \notin [x_{\alpha}]_{\Theta}$  when either  $x \neq y$ , or  $x = y$  and  $\beta > \alpha$ . In the first case, there exist fuzzy open nbds  $U$  and  $V$  of  $y_{\beta}$  and  $x_{\alpha}$  respectively such that  $U \not\# V$  and then  $C1U \not\# V$  (by Lemma 3.2). Then  $U$  is an open q-nbd of  $y_{\beta}$  with  $C1U \not\# x_{\alpha}$  so that  $y_{\beta} \notin [x_{\alpha}]_{\Theta}$ . In the second case, there exist a fuzzy open nbd  $U$  of  $x_{\alpha}$  and an open q-nbd  $V$  of  $y_{\beta}$  such that  $U \not\# V$ . Then  $C1V \not\# U$  so that  $C1V \not\# x_{\alpha}$  and hence  $y_{\beta} \notin [x_{\alpha}]_{\Theta}$ . Finally, for two distinct points  $x, y$  of  $X$ , there exist fuzzy open nbds  $U$  of  $x_{\alpha}$  and  $V$  of  $y_{\beta}$  such that  $U \not\# V$  and hence  $C1U \not\# V$ , i.e.,

$y_1 \in V \leq 1 - C1U$ . Then  $(1 - C1U)(y) = 1 \Rightarrow (C1U)(y) = 0 \Rightarrow y \notin (C1U)_\Theta$ . Conversely, let  $x_\alpha$  and  $y_\beta$  be two distinct fuzzy points in  $X$ .

CASE I. Let  $x \neq y$ . First suppose that at least one of  $\alpha$  and  $\beta$  is less than 1, say  $\alpha < 1$ . Then there exists  $\lambda > 0$  such that  $\alpha + \lambda < 1$ . Now  $x_\lambda \notin [y_\beta]_\Theta$  and hence there exists a fuzzy open nbd  $U$  of  $y_\beta$  such that  $x_\lambda \notin [U]_\Theta$  (by Theorem 2.6). Then  $U \not\subseteq C1V$ , for an open q-nbd  $V$  of  $x_\lambda$ . Since  $V(x) > 1 - \lambda > \alpha$ ,  $V$  and  $U$  are fuzzy open nbds of  $x_\alpha$  and  $y_\beta$  respectively such that  $U \not\subseteq V$ .

Next, suppose  $\alpha = \beta = 1$ . By hypothesis, there exists a fuzzy open nbd  $U$  of  $x_1$  such that  $(C1U)(y) = 0$ . Then  $(1 - C1U)$  is a fuzzy open nbd of  $y_1$  such that  $U \not\subseteq (1 - C1U)$ .

CASE II. Let  $x = y$ . Suppose  $\alpha < \beta$ . Then  $y_\beta \notin [x_\alpha]_\Theta$  and so  $y_\beta \notin [U]_\Theta$ , for some fuzzy open nbd  $U$  of  $x_\alpha$ . Then for an open q-nbd  $V$  of  $y_\beta$ ,  $C1V \not\subseteq U$  and hence  $V \not\subseteq U$ .

DEFINITION 3.4. [6] An fts  $X$  is said to be:

- (a) fuzzy regular (semi-regular) iff for each fuzzy point  $x_\alpha$  in  $X$  and each open q-nbd  $U$  of  $x_\alpha$ , there exists an open q-nbd  $V$  of  $x_\alpha$  such that  $C1V \leq U$  (resp.  $\text{Int } C1V \leq U$ );
- (B) fuzzy almost regular iff for each fuzzy point  $x_\alpha$  in  $X$  and each regularly open q-nbd  $U$  of  $x_\alpha$ , there exists a regularly open q-nbd  $V$  of  $x_\alpha$  such that  $C1V \leq U$ .

THEOREM 3.5. An fts  $X$  is:

- (a) fuzzy regular iff for any fuzzy set  $A$  in  $X$ ,  $C1A = [A]_\Theta$ ;
- (b) fuzzy semi-regular iff  $[A]_\delta = C1A$ , for any fuzzy set  $A$  in  $X$ .

PROOF. Let  $X$  be fuzzy regular. For any fuzzy set  $A$  in  $X$  it is always true that  $C1A \leq [A]_\Theta$ . Now, let  $x_\alpha$  be a fuzzy point in  $X$  such that  $x_\alpha \in [A]_\Theta$  and let  $U$  be any open q-nbd of  $x_\alpha$ . Then by fuzzy regularity of  $X$ , there exists an open q-nbd  $V$  of  $x_\alpha$  such that  $C1V \leq U$ . Now,  $x_\alpha \in [A]_\Theta \Rightarrow C1V \not\subseteq A \Rightarrow U \not\subseteq A \Rightarrow x_\alpha \in C1A$ . Thus  $[A]_\Theta = C1A$ .

Conversely, let  $x_\alpha$  be a fuzzy point in  $X$  and  $U$  an open q-nbd of  $x_\alpha$ . Then  $x_\alpha \notin (1 - U) = C1(1 - U) = [1 - U]_\Theta$ . Thus there exists an open q-nbd  $V$  of  $x_\alpha$  such that  $C1V \not\subseteq (1 - U)$  and then  $C1V \leq U$ . Hence  $X$  is fuzzy regular. (b) Similar to (a) and is omitted.

COROLLARY 3.6. In a fuzzy regular space  $(X, T)$ , a fuzzy closed set is fuzzy  $\Theta$ -closed, and hence for any fuzzy set  $A$  in  $X$ ,  $[A]_\Theta$  is fuzzy  $\Theta$ -closed.

LEMMA 3.7. For any fuzzy semi-open set  $A$  in  $X$ ,  $[A]_\delta = C1A$ .

PROOF. It suffices to show that  $[A]_\delta \leq C1A$ . Let  $x_\alpha \notin C1A$ . Then there exists an open q-nbd  $V$  of  $x_\alpha$  such that  $V \not\subseteq A$ . Then  $\text{Int } C1V \leq \text{Int } C1(1 - A) = 1 - C1 \text{Int } A \leq 1 - A$  (since  $A$  is fuzzy semi-open). Thus  $\text{Int } C1V \not\subseteq A$  and consequently,  $x_\alpha \notin [A]_\delta$ .

THEOREM 3.8. An fts  $X$  is fuzzy almost regular iff  $[A]_\Theta = C1A$ , for every fuzzy semi-open set  $A$  in  $X$ .

PROOF. Let  $X$  be fuzzy almost regular and  $A$  any fuzzy semi-open set in  $X$ . It is enough to show that  $[A]_\Theta \leq C1A$ . Suppose  $x_\alpha \notin C1A$ . By Lemma 3.7, there exists an open q-nbd  $V$  of  $x_\alpha$  such that  $\text{Int } C1V \not\subseteq A$ . Since  $X$  is fuzzy almost regular, there is a fuzzy regularly open set  $U$  such that  $x_\alpha U \leq C1U \leq \text{Int } C1V \leq 1 - A$ . Then  $C1U \not\subseteq A$  and hence  $x_\alpha \notin [A]_\Theta$ . Conversely, let  $U$  be any fuzzy regularly open q-nbd of a fuzzy point  $x_\alpha$ . Then  $x_\alpha \notin 1 - U = C1(1 - U) = [1 - U]_\Theta$ , since a fuzzy regularly closed set is fuzzy semi-open. Hence, there is an open q-nbd  $V$  of  $x_\alpha$  such that  $C1V \not\subseteq (1 - U)$ . Since  $V \leq \text{Int } C1V$ ,  $\text{Int } C1V$  is a regularly open q-nbd of  $x_\alpha$  such that  $C1 \text{Int } C1V = C1V \leq U$  and  $X$  is fuzzy almost regular.

THEOREM 3.9. In an fts  $X$ , the following statements are equivalent:

- (a) For any fuzzy open set  $A$  in  $X$ ,  $[[A]_\Theta]_\Theta = [A]_\Theta$ .
- (b) For any fuzzy set  $A$  in  $X$ ,  $[[A]_\Theta]_\Theta = [A]_\Theta$ .
- (c) For any fuzzy set  $A$  in  $X$ ,  $[A]_\Theta = [A]_\delta$ .
- (d)  $X$  is fuzzy almost regular.

PROOF. (a)  $\Rightarrow$  (d): We first show that for any fuzzy regularly closed set  $F$  in  $X$ ,  $F = [F]_\Theta$ . In fact,  $F$  being fuzzy regularly closed,  $F = C1U$ , for some fuzzy open set  $U$ . Now,  $[F]_\Theta = [C1U]_\Theta = [[U]_\Theta]_\Theta$  (since  $U$  is fuzzy open)  $= [U]_\Theta = C1U = F$ . Next, let  $x_\alpha$  be a fuzzy point in  $X$  and  $A$  any fuzzy

regularly open set in  $X$  with  $x_\alpha qA$ . Then  $x_\alpha \notin (1-A) = [1-A]_\Theta$ , since  $(1-A)$  is fuzzy regularly closed. Hence, there exist a fuzzy open set  $V$  such that  $x_\alpha qV$ , but  $C1V \not q(1-A)$ . Let  $W = \text{Int } C1V$ . Then  $x_\alpha qW$ , and  $C1W = C1V \not q(1-A)$ . Thus,  $W$  is a regularly open q-nbd of  $x_\alpha$  such that  $C1W \leq A$ . Hence,  $X$  is fuzzy almost regular.

(d)  $\Rightarrow$  (c): For any fuzzy set  $A$ , it is clear that  $[A]_\delta \leq [A]_\Theta$ . Now, let  $x_\alpha \in [A]_\Theta$  and  $U$  an open q-nbd of  $x_\alpha$ . Then  $x_\alpha q \text{Int } C1U$ . By (d), there exists a regularly open q-nbd  $V$  of  $x_\alpha$  such that  $C1V \leq \text{Int } C1U$ . Now,  $x_\alpha \in [A]_\Theta \Rightarrow C1V qA \Rightarrow \text{Int } C1U qA \Rightarrow x_\alpha \in [A]_\delta$ .

(c)  $\Rightarrow$  (b):  $[[A]_\Theta]_\Theta = [[A]_\delta]_\Theta = [[A]_\delta]_\delta = [A]_\delta = [A]_\Theta$ .

(b)  $\Rightarrow$  (a): Obvious.

DEFINITION 3.10. A function  $f: X \rightarrow Y$  from an fts  $(X, T)$  to another fts  $(Y, T_1)$  is called

- (i) fuzzy weakly continuous [10] iff for each fuzzy open set  $A$  in  $Y$ ,  $f^{-1}(A) \leq \text{Int } f^{-1}(C1A)$ .
- (ii) fuzzy  $\Theta$ -continuous [6] iff for each fuzzy point  $x_\alpha$  in  $X$  and each open q-nbd  $V$  of  $x_\alpha$ ,  $f(C1U) \leq C1V$ , for some open q-nbd  $U$  of  $x_\alpha$ .

LEMMA 3.11. Let  $f: X \rightarrow Y$  be a function. Then for a fuzzy set  $B$  in  $Y$ ,  $f(1 - f^{-1}(B)) \leq 1 - B$ , where equality holds if  $f$  is onto.

PROOF. Let  $y \in Y$ . If  $f^{-1}(y) = \emptyset$ , then  $[f(1 - f^{-1}(B))](y) = 0 \leq (1 - B)(y)$ . If  $f^{-1}(y) \neq \emptyset$ , then  $[f(1 - f^{-1}(B))](y) = \text{Sup}_{x \in f^{-1}(y)} [1 - f^{-1}(B)](x) = \text{Sup}_{x \in f^{-1}(y)} \{1 - B(f(x))\}$

$$= \text{Sup}_{x \in f^{-1}(y)} \{1 - B(y)\} = 1 - B(y) = (1 - B)(y).$$

If  $f$  is onto, then for each  $y \in Y$ ,  $f^{-1}(y) \neq \emptyset$ , and hence we have  $f(1 - f^{-1}(B)) = 1 - B$ .

THEOREM 3.12. A function  $f: X \rightarrow Y$  is:

- (a) fuzzy weakly continuous iff  $f(C1U) \leq [f(U)]_\Theta$ , for each fuzzy set  $U$  in  $X$ .
- (b) fuzzy  $\Theta$ -continuous iff  $f([A]_\Theta) \leq [f(A)]_\Theta$ , for any fuzzy set  $A$  in  $X$ .

PROOF. (a) Let  $f$  be a fuzzy weakly continuous and  $U$  any fuzzy set in  $X$ . Suppose  $x_\alpha \in C1U$ . It is enough to show that  $f(x_\alpha) \in [f(U)]_\Theta$ . Let  $A$  be any open q-nbd of  $f(x_\alpha)$ . Then  $f^{-1}(A) q x_\alpha$ . By fuzzy weak continuity of  $f$ ,  $f^{-1}(A) \leq \text{Int } f^{-1}(C1A)$  and hence  $\text{Int } f^{-1}(C1A)$  is an open q-nbd of  $x_\alpha$ . Since  $x_\alpha \in C1U$ , we have  $\text{Int } f^{-1}(C1A) q U$ . Then  $f^{-1}(C1A) q U$  and hence  $C1A q f(U)$ . Thus  $f(x_\alpha) \in [f(U)]_\Theta$ .

Conversely, for any fuzzy open set  $U$  in  $Y$ ,  $f(1 - \text{Int } f^{-1}(C1U))$

$$= f(C1(1 - f^{-1}(C1U))) \leq [f(1 - f^{-1}(C1U))]_\Theta \leq [1 - C1U]_\Theta$$

(by Lemma 3.11)

$$= C1(1 - C1U) = 1 - \text{Int } C1U \leq 1 - U \Rightarrow f(1 - \text{Int } f^{-1}(C1U)) \not q U \Rightarrow 1 - \text{Int } f^{-1}(C1U) \not q f^{-1}(U) \\ \Rightarrow f^{-1}(U) \leq \text{Int } f^{-1}(C1U)$$

Hence  $f$  is fuzzy weakly continuous.

(b) Let the condition hold. For any fuzzy point  $x_\alpha$  in  $X$  and any open q-nbd  $A$  of  $f(x_\alpha)$  in  $Y$ , we have by Lemma 3.11,  $f(1 - f^{-1}(C1A)) \leq 1 - C1A$ . Thus,  $C1A \not q f(1 - f^{-1}(C1A))$  so that  $f(x_\alpha) \notin [f(1 - f^{-1}(C1A))]_\Theta$ . By hypothesis,  $f(x_\alpha) \notin f([1 - f^{-1}(C1A)]_\Theta)$  and hence  $x_\alpha \notin [1 - f^{-1}(C1A)]_\Theta$ . Then there is an open q-nbd  $V$  of  $x_\alpha$  such that  $C1V \not q (1 - f^{-1}(C1A))$  and hence  $f(C1V) \leq f f^{-1}(C1A) \leq C1A$ . Thus  $f$  is fuzzy  $\Theta$ -continuous.

The converse part was proved in [6].

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