

A FLAG TRANSITIVE PLANE OF ORDER 49 AND ITS TRANSLATION COMPLEMENT

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ABSTRACT. The translation complement of the flag transitive plane of order 49 [Proc. Amer. Math. Soc. 32 (1972), 256-262] constructed by Rao is computed. It is shown that the flag transitive group itself is the translation complement and it is a solvable group of order 600.

1. INTRODUCTION.

Rao [1] has constructed a non-Desarguesian translation plane π of order 49 and exhibited a collineation group that is transitive on the distinguished points of π . In this paper we have computed the translation complement G of π and shown that the flag transitive collineation group is the translation complement of π . Further, G is a solvable group of order 600.

2. DESCRIPTION OF π AND ITS FLAG TRANSITIVE COLLINEATION GROUP.

Throughout this paper F , (a,b,c,d) , $\det M$ and $d.p.$ denote the finite field $GF(7)$, the two by two matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of M and the distinguished point respectively.

The translation plane π under study was constructed through a 1-spread set ℓ over F (see Lemma 3.1 of [1, p.258]) whose elements $L_i, 1 < i < 49$, were generated by

$$L_{i+1} = (A+L_iP)^{-1}A$$

where $A = (1,0;1,4), P = (5,6;0,6), L_1 = (0,0;0,0)$ and $L_{25} = (2,0;5,3)$ and it may be constructed as follows:

Let $V_i = \{(x,y) \mid x, y \in Fx, y = xL_i\}$ and $V_0 = \{(0,y) \mid 0 = (0,0), y \in Fx\}$. The incidence structure whose points are the vectors of $F^4 = V$ and whose lines are $V_i, 0 < i < 49$, and their right cosets in the additive group of V with inclusion as the incidence relation is the translation plane π .

Any nonsingular linear transformation on V induces a collineation of π if and only if the linear transformation permutes the subspaces $V_i, 0 < i < 49$, among themselves. Lemma 4.1 of [1] and Theorem 1 of [2] are now used in this paper to compute the collineations of π . Rao has shown that the linear transformations

$$R = \begin{pmatrix} A & A \\ P & 0 \end{pmatrix} \text{ and } S = \begin{pmatrix} U & V \\ W & Z \end{pmatrix}$$

where $U = (0,3;5,0), V = (3,5;0,5), W = (4,5;2,5)$ and $Z = (6,0;2,1)$ on V induce collineations on π and their actions on the set of d.p.s. of π are

$$R: (0,1,2, \dots, 24) (25,26,27, \dots, 49)$$

$$S: (0,38) (1,45,24,31) (2,27,23,49) (3,34,22,42) (4,41,21,35) \\ (5,48,20,28) (6,30,19,46) (7,37,18,39) (8,44,17,32) \\ (9,26,16,25) (10,33,15,43) (11,40,14,36) (12,47,13,29).$$

From the actions of R and S , it is clear that the group $G' = \langle R, S \rangle$ is transitive on the set of d.p.s of π and consequently G' is flag transitive group of π .

3. SPREAD SETS OF π AND SOME OF THEIR PROPERTIES.

We say that a spread set over F of π has a det. structure $(a_1, a_2, a_3, a_4, a_5, a_6)$ if the number of matrices of the spread set which are of determinant i is $a_i, 1 < i < 6$.

It may be noted that the spread set ℓ of π was constructed by taking V_0, V_1 and V_2 as the fundamental subspaces ($x = y, y = 0$ and $y = x$ respectively) and the det. structure of ℓ is $(9,9,6,8,8,8)$. We now construct another 1-spread set ℓ' from

ℓ of π with the fundamental subspaces V_0, V_{38} and V_4 , since the spread set ℓ is not amenable for easy computations and we study some properties of ℓ' and det. structures of certain matrix representative sets of π . This information is useful in the computation of the translation complement G of π .

Let T be a 4×4 matrix given by $T = \begin{pmatrix} I & C \\ 0 & D \end{pmatrix}$ where $C = (5,6;1,3), D = (4,4;5,0), I = (1,0; 0,1)$ and 0 is the 2×2 zero matrix. Define for each $L_i \in \ell, 1 < i < 49$,

$$M_i = C + L_i D$$

Let $\ell' = \{M_i \mid 1 \leq i \leq 49\}$. The matrices M_i , $1 \leq i \leq 49$ are listed in table 3.1. The entry a,b under the heading C.P. of M_i indicates that the matrix M_i has the characteristic polynomial $\lambda^2 + a\lambda + b$. It may be noted that the det. structure of ℓ' is (4,12,6,12,6,8). If $A, B \in GL(2, F)$ then the det. structure of $A^{-1}\ell'B$ is

- (4,12,6,12,6,8) if $\det A^{-1}B = 1$
- (12,4,6,12,8,6) if $\det A^{-1}B = 2$
- (6,6,4,8,12,12) if $\det A^{-1}B = 3$
- (12,12,8,4,6,6) if $\det A^{-1}B = 4$
- (6,8,12,6,4,12) if $\det A^{-1}B = 5$
- (8,6,12,6,12,4) if $\det A^{-1}B = 6$

We say that the above det. structures are the allied det. structures of ℓ' .

Table 3.1

i	M_i	C.P. of M_i	i	M_i	C.P. of M_i
1	(5,6;1,3)	6,2	26	(2,5;0,6)	6,5
2	(2,3;6,3)	2,2	27	(4,2;3,6)	4,4
3	(1,5;1,0)	6,2	28	(5,5;4,3)	6,2
4	(1,0;0,1)	5,1	29	(6,2;2,0)	1,3
5	(0,2;5,4)	3,4	30	(4,5;0,5)	5,6
6	(4,0;1,5)	5,6	31	(1,6;6,5)	1,4
7	(2,4;5,6)	6,6	32	(6,6;3,1)	0,2
8	(0,1;3,2)	5,4	33	(5,4;2,1)	1,4
9	(5,2;0,2)	0,3	34	(6,1;0,3)	5,4
10	(3,3;2,5)	6,2	35	(1,3;4,6)	0,1
11	(4,6;2,2)	1,3	36	(5,3;3,0)	2,5
12	(0,4;4,5)	2,5	37	(3,2;5,3)	1,6
13	(3,5;0,4)	0,5	38	(0,0;0,0)	
14	(2,1;2,6)	6,3	39	(4,1;5,1)	2,6
15	(0,6;2,4)	3,2	40	(0,5;6,1)	6,5
16	(6,4;4,2)	6,3	41	(3,4;1,4)	0,1
17	(0,3;1,1)	6,4	42	(6,3;5,2)	6,4
18	(5,0;6,4)	5,6	43	(4,3;4,4)	6,4
19	(2,0;3,3)	2,6	44	(1,2;5,5)	1,2
20	(4,4;6,0)	3,4	45	(3,6;4,0)	4,4
21	(2,2;4,1)	4,1	46	(2,6;1,6)	6,6
22	(3,1;3,4)	0,2	47	(6,5;6,2)	6,3
23	(5,1;5,0)	2,2	48	(1,4;2,3)	3,2
24	(1,1;3,5)	1,2	49	(3,0;6,6)	5,4
25	(6,0;1,2)	6,5			

The planes associated with ℓ and ℓ' are isomorphic and the isomorphism is given by T. Without any loss of generality we take the plane associated with ℓ' as π since ℓ' is one of the spread sets of π . The collineations R and S now become the collineations α and δ of π , where $\alpha = T^{-1}RT$, $\delta = T^{-1}ST$. The actions of α and δ on the set of d.p.s of π are same as R and S. Therefore $\langle \alpha, \delta \rangle$ is the flag transitive

collineation group of π . Since δ is a collineation of π , $\delta^2 = \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}$ where $Q = (2,5;3,5)$, $R = (3,2;5,4)$ is also a collineation of π and its action on the set of d.p.s of π is given by

$$\begin{aligned} \delta^2: & (0)(38)(1,24)(2,23)(3,22)(4,21)(5,20)(6,19)(7,18) \\ & (8,17)(9,16)(10,15)(11,14)(12,13)(25,26)(27,49) \\ & (28,48)(29,47)(30,46)(31,45)(32,44)(33,43)(34,42) \\ & (35,41)(36,40)(37,39) \end{aligned}$$

LEMMA 3.1. $A^{-1}\ell'A = \ell'$, if and only if A is a scalar matrix.

PROOF. If A is a scalar matrix then the lemma follows trivially. Conversely suppose that $A^{-1}\ell'A = \{A^{-1}MA/M \in \ell'\} = \ell'$. From table 3.1. we notice that ℓ' contains M_9 and M_{13} with the characteristic polynomials $\lambda^2 + 3$ and $\lambda^2 + 5$ respectively and no other matrix of ℓ' has these polynomials as the characteristic polynomial. Therefore, we have, $A^{-1}M_9A = M_9$ and $A^{-1}M_{13}A = M_{13}$. Taking $A = (a,b;c,d)$ and solving the simultaneous equations obtained from $M_9A = AM_9$ and $M_{13}A = AM_{13}$ we get $b = c = 0$ and $a = d$. Hence the lemma.

LEMMA 3.2. Let $M_k \in \ell'$. The spread sets ℓ' and $\ell'M_k^{-1}$ are conjugate if $k=4, 21$ and are not conjugate otherwise.

PROOF. The first part of the lemma follows from lemma 3.1 and the collineation δ^2 when $k=4$ and $k = 21$ respectively. If ℓ' and $\ell'M_k^{-1}$ are conjugate then their det. structures must be same and this is possible if the $\det M_k = 1$. Therefore ℓ' and $\ell'M_k^{-1}$ are not conjugate if $\det M_k \neq 1$. The matrices of ℓ' which are of determinant 1 are M_k , $k=4,21,35$ and 41. If $k=35$ then $M_{11}M_k^{-1} \in \ell'M_k^{-1}$ and its characteristic polynomial is $\lambda^2 + 4\lambda + 3$. The spread sets ℓ' and ℓ',M_{35}^{-1} are not conjugate since ℓ' does not contain a matrix with the characteristic polynomial $\lambda^2 + 4\lambda + 3$. We reject $k=41$ by observing the characteristic polynomial of $M_3M_k^{-1}$ and using the same argument as in the previous case. The lemma now follows.

Let $M_k \in \ell'$. The det. structures of $\ell' - M_k = \{M - M_k \mid M \in \ell'\}$ are computed and are furnished in the table 3.2 for specified values of k . This information is useful in the sequel.

Table 3.2.

k	The det. structure of $\ell' - M_k$	k	The det. structure of $\ell' - M_k$
1	(9,9,6,8,8,8)	25	(6,10,8,4,14,6)
2	(6,7,11,7,11,6)	27	(6,11,10,11,4,6)
3	(7,7,8,6,10,10)	28	(5,8,5,9,12,9)
4	(6,10,8,6,10,8)	29	(6,4,14,10,8,6)
5	(13,4,5,13,8,5)	30	(12,5,6,7,11,7)
6	(8,6,10,6,10,8)	31	(7,10,6,9,5,11)
7	(5,6,11,5,10,11)	32	(12,7,11,5,6,7)
8	(7,7,10,12,6,6)	33	(8,5,5,9,9,12)
9	(6,6,10,8,9,9)	34	(5,9,8,16,6,4)
10	(2,10,8,10,8,10)	35	(16,9,4,5,6,8)
11	(9,2,10,9,8,10)	36	(11,6,10,11,6,4)
12	(10,6,7,10,8,7)	37	(10,9,11,7,5,6)

4. THE TRANSLATION COMPLEMENT G OF π .

Let G_0 be the group of all collineations of π that fix the d.p. 0; $G_{0,38}$ be the group of all collineations of π that fix the d.p.s. 0 and 38 and $G_{0,38,4}$ be the group of all collineations that fix the d.p.s 0, 38 and 4.

LEMMA 4.1. $G_{0,38,4}$ is generated by scalar collineations and it is of order 6.

PROOF. Any collineation $\sigma \in G_{0,38,4}$ is of the form $\sigma = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ for some $A \in GL(2,F)$, satisfying the condition that for every matrix $m \in \ell'$ there exists a matrix $N \in \ell'$ such that $A^{-1} m A = N$. That is, $A^{-1} \ell' A = \ell'$. By lemma 3.1 we have $A = (a,0;0,a)$, $a \in F$, $a \neq 0$ and σ always induces a collineation of π fixing all the d.p.s. of π . Such a collineation σ is called a scalar collineation. If a is a generator of F then $G_{0,38,4} = \langle \sigma \rangle$ and it is of order 6. Hence the lemma.

LEMMA 4.2. $G_{0,38} = \langle \delta^2 \rangle$ and it is of order 12.

PROOF. Any collineation $\beta \in G_{0,38}$ is of the form $\beta = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ for some $A, B \in GL(2,F)$. Further A and B must satisfy the condition that for each matrix $M \in \ell'$ there exists a matrix $N \in \ell'$ such that $A^{-1} M B = N$. Taking $M = M_4$ we get a condition that $A^{-1} B \in \ell'$. Let $A^{-1} B = M_k$ for some k , $1 < k < 49$, $k \neq 38$. Then we obtain that the spread sets ℓ' and $\ell' M_k^{-1}$ are conjugate. By lemma 3.2 we have $k=4$ and 21. Therefore every collineation $\beta \in G_{0,38}$ either fixes the d.p. 4 or maps the d.p. 4 onto the d.p. 21 and hence β either fixes the d.p. 4 or interchanges the d.p.s 4 and 21. Since δ^2 is a collineation of $G_{0,38}$ interchanging the d.p.s 4 and 21, we have

$$G_{0,38} = G_{0,38,4} \quad G_{0,38,4} \delta^2 = \langle \sigma, \delta^2 \rangle = \langle \delta^2 \rangle$$

since $\langle \sigma \rangle < \delta^2 \rangle$. Further, $|G_{0,38}| = 2|G_{0,38,4}| = 12$. Hence the lemma.

LEMMA 4.3. $G_0 = G_{0,38}$.

PROOF. If $\gamma \in G_0$ and maps the d.p. 38 onto the d.p.k then δ is of the form $\delta = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ for some A, B and $D \in GL(2,F)$, satisfying the condition that for each matrix $M \in \ell'$ there exists a matrix $N \in \ell'$ such that $A^{-1}(B + MD) = N$ and $A^{-1}B = M_k$. That is, for every matrix $M \in \ell'$ there exists a matrix $N \in \ell'$ such that $N - M_k = A^{-1}MD$. Suppose that the collineation γ maps the d.p.k onto the d.p.k' then $\delta^{-2} \gamma \delta^2 \in G_0$ and maps the

d.p. 38 onto the d.p.k'. Observing the action of δ^2 on the d.p.s, it is therefore enough if we take the values of k from $\{1, 25, j \mid 1 < i < 11, 27 < j < 37\}$. For these values of k the det. structure of $\ell' - M_k$ is same as the det. structure of $A^{-1}\ell' D$. That is, for these values of k the det. structure of $\ell' - M_k$ must coincide with one of the allied det. structure of ℓ' . This is not possible (see table 3.2). If $\gamma \in G_0$ then γ fixes the d.p. 38 and therefore $\gamma \in G_{0,38}$. Hence the lemma.

THEOREM 4.4. The translation complement G of π is $\langle \alpha, \delta \rangle$ and it is a solvable group of order 600.

PROOF. Since $\langle \alpha, \delta \rangle$ is transitive on the set of all d.p.s

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of π , $G = \prod_{i=0}^{49} G_0 \alpha_i$ where α_i is a collineation of π which maps the d.p. 0 onto the d.p. i and it may be taken from $\langle \alpha, \delta \rangle$. The order of $G = 50 \times (\text{order of } G_0) = 600$. In view of lemma 4.3, $G = \langle \alpha, \delta \rangle$. Notice that $\delta^{-1} \alpha \delta = \alpha^7$ and $G \langle \alpha \rangle \{e\}$ is a solvable series of G. Therefore G is a solvable group and hence the theorem.

It is interesting to note that the flag transitive group of π itself is the translation complement of π and the two flag transitive planes of order 25 constructed by Foulser [3] also possess this property [4,5].

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