

## FROM PATHS TO STARS

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**ABSTRACT.** The number of cycles in the complement  $T'$  of a tree  $T$  is known to increase with the diameter of the tree. A similar question is raised and settled for the number of complete subgraphs in  $T'$  for a special class of trees via Fibonacci numbers. A structural characterization of extremal trees is also presented.

**KEY WORDS AND PHRASES.** Cycles, Complete subgraphs, Trees.

**1980 AMS SUBJECT CLASSIFICATION CODE.** 05-C.

### 1. INTRODUCTION.

Among all trees  $T$  of order  $n$ , the number  $c(T')$  of all cycles in the complement  $T'$  and the structural characterization of those trees which optimize  $c(T')$  have been dealt with in [1,2]. The same problem was solved in [3] for the number  $i(T')$  of all complete subgraphs in the complement of an arbitrary  $T$ . It turns out that among all  $n$ -trees  $T$ , the path  $P_n$  ( $n \geq 9$ ) has both the maximum number of cycles [1] and the minimum number of complete subgraphs in its complement [3]. The star  $S_n$  also maximizes both  $c(T')$ ,  $5 \leq n \leq 8$  [1] and  $i(T')$  for  $n \geq 4$  [2]. It minimizes  $c(T')$  for  $n \geq 9$ .

The problem of characterizing the  $n$ -vertex trees  $T$  for extremal values with respect to  $c(T')$  or  $i(T')$  loses some structural significance in the generality of  $Y$ . Suppose we consider a class of trees which keep out all paths and stars, for example, the class  $\mathcal{F}_3$  of all those trees  $T_3$  having exactly three endvertices. What structural similarities between  $c(T'_3)$  and  $i(T'_3)$  are inherited from  $c(T')$  and  $i(T')$ ? We recall that the diameter of a graph  $G$  is the maximum distance  $d(u, v)$  taken over all pairs of vertices  $u, v$  in  $G$ . The following theorem [1, p.93] relates  $c(T')$  with the diameter of  $T$ .

**THEOREM 1.** For each  $n \geq 6$  and every tree  $T$  of order  $n$  and diameter  $d$ ,  $4 \leq d \leq n - 2$ , there is a tree  $T_1$  of order  $n$  and at least diameter  $d + 1$  such that  $c(T') < c(T'_1)$ .

### 2. TREES WITH THREE ENDVERTICES.

Utilizing enumerative techniques [2] we conclude that among all trees  $T_3$  of order  $n$ , the

tree with the smallest number of cycles in its complement is a tree with the smallest diameter as shown in Figure 1. Moreover, the tree  $T_3$

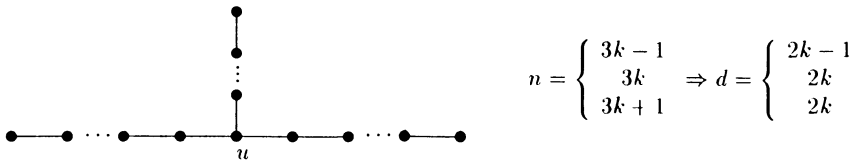


Figure 1. A tree with three endvertices and minimum  $c(T'_3)$ .

with the largest number of cycles in its complement is a tree with the largest diameter as shown in Figure 2.

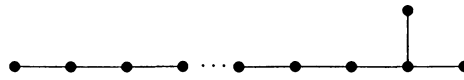


Figure 2. A tree with maximum  $c(T'_3)$ .

So, among all trees with three endvertices, of order  $n$ , the direct relationship between  $c(T'_3)$  and the diameter of  $T_3$  is inherited from the class of all trees, i.e.,  $\min_{\mathcal{F}_3} c(T'_3)$  and  $\max_{\mathcal{F}_3} c(T'_3)$  are still associated with the smallest and the largest diameters of  $T_3$ , respectively. Can we make the same claim about  $i(T'_3)$ , the number of complete subgraphs in the complement of  $T_3$ ? We recall that when  $T$  is arbitrary,  $i(T')$  is maximum when the diameter of  $T$  is minimum ( $T$  is a star) and  $i(T')$  is minimum when the diameter is maximum ( $T$  is a path). Does this relationship between  $i(T')$  and the diameter of  $T$  remain true when  $T$  is restricted to  $\mathcal{F}_3$ ? To this end, we need the concept of a Fibonacci number  $f(G)$  of a graph  $G$ .

According to [4, p. 45], the total number of subsets of  $\{1, 2, 3, \dots, n\}$  such that no two elements are adjacent is  $F_{n+1}$ , where  $F_n$  is the  $n$ th Fibonacci number, which is defined by

$$F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}, \quad n \geq 2.$$

The sequence  $\{1, 2, 3, \dots, n\}$  can be regarded as the vertex set of the path  $P_n$ . This definition covers the empty graph also; so,  $f(G) = i(G')$ . We note that  $i(P'_0) = 1, i(P'_1) = 2, i(P'_2), \dots, i(P'_n) = F_{n+1}$ .

### 3. MAIN RESULTS

If  $T_3$  is a tree with three endvertices, then it has a unique vertex  $u$  of degree three. We count  $i(T'_3)$  [5] by considering two disjoint sets of complete subgraphs of  $T'_3$ , say  $S_1$  and  $S_2$ , where  $S_1$  is the set of those complete subgraphs not containing the vertex  $u$ , and  $S_2$  consists of those that do contain  $u$ . Let  $v_1, v_2$  and  $v_3$  be the three vertices adjacent to  $u$  in  $T_3$ . We have

$$i(T'_3) = |S_1| + |S_2| = i(T_3 - u)' + i(T_3 - v_1 - v_2 - v_3)'.$$

If  $n = 3k + 1, T_3 - u$  is a union of three disjoint paths on  $3k$  vertices, where  $T_3 - v_1 - v_2 - v_3$

is also a union of three disjoint paths on  $3k - 3$  vertices together with the isolated vertex  $u$  (see Figure 1). The following theorem on Fibonacci numbers shows that  $i(T'_3)$  is minimized and maximized by the trees in Figures 3a and 3b, respectively. This shows

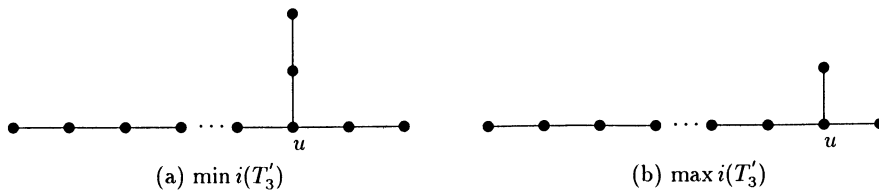


Figure 3. Extremal trees in  $\mathcal{F}_3$ .

that the inverse relationship between  $i(T')$  and the diameter of  $T$  is not inherited in the class of trees  $\mathcal{F}_3$  with exactly three endvertices.

**THEOREM 2.** Let  $n$  be an integer  $\geq 7$ . Then among all summands  $r_1, s_1, t_1$  and  $r_2, s_2, t_2$  satisfying (i)  $r_1 + s_1 + t_1 = n + 2$ , (ii)  $r_1 = r_2 + 1, s_1 = s_2 + 1, t_1 = t_2 + 1$ , and  $r_1, s_1, t_1 \geq 2$  we have

the sum of the two products  $F_{r_1}F_{s_1}F_{t_1} + F_{r_2}F_{s_2}F_{t_2}$  of the Fibonacci numbers  $F_{r_1}, F_{s_1}, F_{t_1}$  and  $F_{r_2}, F_{s_2}, F_{t_2}$  is

(a) minimum if  $r_1 = s_1 = 3$  and  $t_1 = n - 4$  and

(b) maximum if  $r_1 = s_1 = 2$  and  $t_1 = n - 2$ .

**PROOF.** The order of growth of  $F_n$  is governed by the golden ratio  $\tau = (1 + \sqrt{5})/2$ . Moreover,  $F_n \approx c\tau^n$  where  $c = \tau/\sqrt{5}$  and  $F_{n+1} \approx \tau F_n$ . We have  $r_1 + s_1 + t_1 = n + 2$  and  $r_2 + s_2 + t_2 = n - 1$ , and for large  $n$ ,

$$\begin{aligned} F_{r_1}F_{s_1}F_{t_1} + F_{r_2}F_{s_2}F_{t_2} &\approx (\tau F_{r_2})(\tau F_{s_2})(\tau F_{t_2}) + F_{r_2}F_{s_2}F_{t_2} = (1 + \tau^3)F_{r_2}F_{s_2}F_{t_2} \\ &\approx (5.236068 \dots)(c^3 \tau^{r_2+s_2+t_2}) \approx 13.59764677 \dots \tau^{n-5} > 13.43181071 \dots \tau^{n-5} \\ &\approx 9F_{n-4} + 4F_{n-5} = F_3F_3F_{n-4} + F_2F_2F_{n-5} \\ &= i(P'_2)i(P'_2)i(P'_{n-5}) + i(P'_1)i(P'_1)i(P'_{n-6}). \end{aligned}$$

That is,  $\min_{\mathcal{F}_3} i(T'_3)$  is realized in Figure 3a.

To prove (b), we note that for the tree in Figure 3b, we have  $i(T'_3) = 4F_{n-2} + F_{n-3} \approx 12.09016992 \dots F_{n-4}$ . On the other hand, for an arbitrary  $T_3$  with  $n$  large enough, we have  $F_{r_1}F_{s_1}F_{t_1} + F_{r_2}F_{s_2}F_{t_2} \approx 13.59764677 \dots \tau^{n-5} \approx 11.61377685 \dots F_{n-4} < 12.09016992 \dots F_{n-4} \approx 4F_{n-2} + F_{n-3}$ . That is,  $\max_{\mathcal{F}_3} i(T'_3)$  is realized in Figure 3b.

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