

RESEARCH NOTES

A NEW PROOF OF A THEOREM ABOUT GENERALIZED ORTHOGONAL POLYNOMIALS

A.McD. MERCER

Department of Mathematics and Statistics
 University of Guelph
 Ontario, Canada

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ABSTRACT. In this note it is shown that a fairly recent result on the asymptotic distribution of the zeros of generalized polynomials can be deduced from an old theorem of G. Polya, using a minimum of orthogonal polynomial theory.

KEY WORDS AND PHRASES. Toeplitz matrices, orthogonal polynomials, asymptotic zero distribution.
MATHEMATICAL REVIEWS SUBJECT CLASSIFICATION. 15 A 18, 26 C 10.

1. INTRODUCTION.

Let $\alpha(x)$ be a distribution function [1] which has associated with it the unique sequence of polynomials $p_n(x) (n = 0, 1, 2, \dots)$ satisfying

$$p_n(x) = \gamma_n x^n + \dots \quad (\gamma_n > 0)$$

and

$$\int_{-\infty}^{+\infty} p_m(x)p_n(x)d\alpha(x) = \delta_{m,n} \quad (m, n = 0, 1, 2, \dots) \quad (1.1)$$

These polynomials will satisfy the recurrence relation:

$$\left. \begin{aligned} xp_n(x) &= \frac{\gamma_n}{\gamma_{n+1}}p_{n+1}(x) + \alpha_n p_n(x) + \frac{\gamma_{n-1}}{\gamma_n}p_{n-1}(x) \\ p_{-1} &= \gamma_{-1} = 0, \quad p_0(x) = \gamma_0, \quad \alpha_n \in \mathfrak{R}, \quad \gamma_n > 0 \quad (n = 0, 1, 2, \dots) \end{aligned} \right\} \quad (1.2)$$

Conversely, a theorem due to Favard [2] states that if a sequence of polynomials satisfies (1.2) then a distribution function $\alpha(x)$ exists such that (1.1) holds. It is also known that the function $\alpha(x)$ is essentially unique [3] whenever the sequences $\left\{ \frac{\gamma_n}{\gamma_{n+1}} \right\}$ and $\{\alpha_n\}$ are both bounded and that a necessary and sufficient condition for this to be the case is that the support of $d\alpha$,

$$\text{supp}(d\alpha) = \{x: \alpha(x - \epsilon) < \alpha(x + \epsilon), \forall \epsilon > 0\}$$

is compact.

Suppose now that in (1.2) $\alpha_n \rightarrow a$ and $\frac{\gamma_n}{\gamma_{n+1}} \rightarrow \frac{b}{2}$. Then one writes $\alpha \in M(a, b)$ and if $b > 0$ there

will be no loss of generality in supposing that $\alpha \in M(0, 1)$ [4], and we shall take this to be the case in all that follows. We now have $\text{supp}(d\alpha)$ as a compact set and we will denote by Δ the smallest compact interval containing it. To elucidate the nature of $\text{supp}(d\alpha)$ we quote the following lemma [4].

LEMMA A. Let $\alpha \in M(0, 1)$ then $\text{supp}(d\alpha)$ can be written as $\text{supp}(d\alpha) = [-1, +1] \cup B$ where B is enumerable and bounded and the only possible limit points of B are ± 1 . Furthermore, if X is the set of all of the zeros of all the $p_n(x)$, then all of the limit points of X belong to $\text{supp}(d\alpha)$.

Once the polynomials $p_n(x)$ have been obtained, one can take a real-valued function f defined on $\text{supp}(d\alpha)$ and suitably restricted, and form the Toeplitz matrix

$$H_n(f) = \left[\int_{\Delta} f(x)p_i(x)p_j(x)d\alpha(x) \right] \quad (i, j = 0, 1, \dots, n - 1)$$

We shall denote the eigenvalues of this symmetric matrix, taken in ascending order, by $x_{k,n}(f)$ ($1 \leq k \leq n$). The following important theorem, with slight weaker hypotheses than are used here, has been proved by P. Nevai in [4]

THEOREM A. Let $\alpha \in M(0, 1)$. Let f be real-valued and $f \in C(\text{supp}(d\alpha))$. Let Γ be a compact interval containing $f(\text{supp}(d\alpha))$ and suppose that $F \in C(\Gamma)$. Then, as $n \rightarrow \infty$

$$\frac{1}{n} \sum_{k=1}^n F(x_{k,n}(f)) \rightarrow \frac{1}{\pi} \int_{-1}^{+1} \frac{F(f(t))}{\sqrt{1-t^2}} dt$$

NOTE. It is not difficult to see that if m and M denote the lower and upper bounds of f on $\text{supp}(d\alpha)$ then

$$m \leq x_{k,n}(f) \leq M \quad (1 \leq k \leq n; n = 1, 2, \dots)$$

We complete this introduction by quoting a theorem which we call Theorem B. It is a special case of a theorem appearing in [5] (see Section 5.2 there).

THEOREM B. Let $f_1 \in C[-1, +1]$ be real-valued and write $g_1(\theta) = f_1(\cos \theta)$. Form the symmetric matrix

$$K_n(f_1) = \left[\frac{1}{\pi} \int_0^{\pi} g_1(\theta) \cos(i-j)\theta d\theta \right] \quad (i, j = 0, 1, \dots, n - 1)$$

Denoting by m_1 and M_1 the lower and upper bounds of f_1 , let $F_1 \in C[m_1, M_1]$. Then as $n \rightarrow \infty$ we have

$$\frac{1}{n} \sum_{k=1}^n F_1(\lambda_{k,n}(f_1)) \rightarrow \frac{1}{\pi} \int_0^{\pi} F_1(g_1(\theta)) d\theta$$

In this $\lambda_{k,n}(f_1)$ are the eigenvalues of $K_n(f_1)$ (all of which lie in $[m_1, M_1]$).

The purpose of this note is to give an alternative proof of Theorem A, showing that it may be deduced from Theorem B using two auxiliary lemmas.

2. AUXILIARY LEMMAS

LEMMA B. Let $\alpha \in M(0, 1)$ and let l be a fixed non-negative integer. Let $f \in C(\text{supp}(d\alpha))$. Then as $n \rightarrow \infty$

$$\int_{\Delta} f(x)p_n(x)p_{n+l}(x)d\alpha(x) \rightarrow \frac{1}{\pi} \int_{-1}^{+1} f(t) \frac{T_l(t)dt}{\sqrt{1-t^2}}$$

where $T_i(t)$ is the Chebychev polynomial.

LEMMA C. Let $A \equiv [a_{i,j}], B \equiv [b_{i,j}] (i, j = 0, 1, \dots)$ be Hermitian matrices whose principal $n \times n$ sections have eigenvalues

$$\alpha_{1,n} \leq \alpha_{2,n} \leq \dots \alpha_{n,n}$$

and

$$\beta_{1,n} \leq \beta_{2,n} \leq \dots \leq \beta_{n,n}$$

For all n let all of these eigenvalues lie in a compact interval $[m_2, M_2]$. Let $k(n) (n = 1, 2, \dots)$ be non-negative integers such that $k(n) = o(n)$ and let

$$\|A_n^{(k)} - B_n^{(k)}\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

where $A_n^{(k)}$, for example, denotes the matrix $[a_{i,j}] (i, j = k, k + 1, \dots, k + n - 1)$ and $\|\cdot\|$ is any matrix norm.

Then as $n \rightarrow \infty$ we have

$$\frac{1}{n} \sum_{k=1}^n F_2(\alpha_{k,n}) - \frac{1}{n} \sum_{k=1}^n F_2(\beta_{k,n}) \rightarrow 0$$

for any $F_2 \in C[m_2, M_2]$.

Proofs of Lemma B are to be found in [4] and [6] while Lemma C is a special case of a result proved in [7].

3. MAIN RESULTS.

We now proceed to prove our result, namely,

THEOREM 1. Theorem B implies Theorem A.

PROOF. $f \in C(\text{supp}(d\alpha))$ and $\text{supp}(d\alpha)$ is closed so we can extend the definition of f to obtain $f_* \in C(\Delta)$ and this can be done in such a way that the bounds of f_* are also m and M . Next, a polynomial q can be found so that $\text{Sup}_\Delta |f_*(x) - q(x)|$ is as small as we please and the bounds of q on Δ are also m and

M . We shall prove the theorem, in the first instance, for such a polynomial q . The virtue of working with q instead of the original f lies in the fact that the two matrices which appear in Theorems A and B will, then, each be banded. This makes Lemma C easy to apply.

Since the bounds of q are the same as those given originally for f , we note that $\Gamma \supset q(\Delta)$. As in Theorem A let $F \in C(\Gamma)$. Now in Theorem B take f_1 to be q and F_1 to be F . In the notation of that theorem $m_1 = \inf_{[-1,+1]} q, M_1 = \sup_{[-1,+1]} q$ and since $[-1, +1] \subset \Delta$ then $F \in C[m_1, M_1]$ (since $[m_1, M_1] \subset [m, M]$).

We deduce from Theorem B that

$$\frac{1}{n} \sum_{k=1}^n F(\lambda_{k,n}(q)) \rightarrow \frac{1}{\pi} \int_0^\pi F(g_2(\theta)) d\theta \tag{3.1}$$

where $g_2(\theta) = q(\cos \theta)$ and in which $\lambda_{k,n}(q)$ are the eigenvalues of

$$K_n(q) = \left[\frac{1}{\pi} \int_0^\pi g_2(\theta) \cos(i-j)\theta d\theta \right] \\ = \left[\frac{1}{\pi} \int_{-1}^{+1} q(t) T_{|i-j|}(t) \frac{dt}{\sqrt{1-t^2}} \right] \quad (i, j = 0, 1, 2, \dots, n-1)$$

We next compare the infinite matrices $K_\infty(q)$ and

$$H_\infty(q) = \left[\int_{\Delta} q(x)p_i(x)p_j(x)d\alpha(x) \right] \quad (i, j = 0, 1, 2, \dots)$$

and we remark, first, that each is banded with bandwidth

$$2(\text{degree } q) + 1 = N \quad (\text{say})$$

We next note that, according to Lemma B,

$$\int_{\Delta} q(x)p_i(x)p_j(x)d\alpha(x) - \frac{1}{\pi} \int_{-1}^{+1} q(x)T_{|i-j|}(x) \frac{dx}{\sqrt{1-x^2}} \rightarrow 0 \tag{3.2}$$

as $i \rightarrow \infty$ for each fixed $|i - j| = 0, 1, 2, \dots, N$. To apply Lemma C we take $k(n) = \lfloor \sqrt{n} \rfloor$ and the matrix norm to be $\|\cdot\|_\infty$. According to (3.2) we will have

$$\|K_n^{(k)}(q) - H_n^{(k)}(q)\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and so we conclude that

$$\frac{1}{n} \sum_{k=1}^n F(\lambda_{k,n}(q)) - \frac{1}{n} \sum_{k=1}^n F(x_{k,n}(q)) \rightarrow 0 \tag{3.3}$$

as $n \rightarrow \infty$. From (3.1) and (3.3) we deduce that, as $n \rightarrow \infty$

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n F(x_{k,n}(q)) &\rightarrow \frac{1}{\pi} \int_0^\pi F(g_2(\theta))d\theta \\ &= \frac{1}{\pi} \int_{-1}^{+1} \frac{F(q(t))}{\sqrt{1-t^2}} dt \end{aligned} \tag{3.4}$$

which completes the proof of Theorem 1 for the polynomial case.

We can now extend (3.4) to the general case using the customary type of approximation argument. Let $\epsilon > 0$ be given. Then, by the uniform continuity of F on Γ , there will be $\delta > 0$ such that

$$|F(\eta_1) - F(\eta_2)| < \epsilon \quad \text{whenever } |\eta_1 - \eta_2| < \delta \quad (\eta_1, \eta_2 \in \Gamma)$$

With $f \in C(\text{supp}(d\alpha))$ as given, we find the polynomial q as previously described so that

$$|f_+(t) - q(t)| < \delta \quad \text{on } \Delta \quad \text{and } m \leq q(t) \leq M \quad \text{on } \Delta$$

We now have

$$|F(f(t)) - F(q(t))| < \epsilon \quad \text{on } [-1, +1] \tag{3.5}$$

Next, consider the matrix

$$H_n(f) = \left[\int_{\Delta} f(x)p_i(x)p_j(x)d\alpha(x) \right] \quad (i, j = 0, 1, \dots, n - 1)$$

and let $\|\cdot\|_2$ denote the spectral norm (= max |eigenvalues|). Since

$$|f_+(t) - q(t)| < \delta \quad \text{on } \Delta$$

and

$$\begin{aligned}
 |x_{k,n}(q) - x_{k,n}(f)| &\leq \|H_n(q) - H_n(f)\|_2 \quad (\text{see [8]}) \\
 &\leq \sup_{\text{supp}(d\alpha)} |q(t) - f(t)| \quad (\text{see [4]}) \\
 &\leq \sup_{\Delta} |q(t) - f_*(t)| < \delta
 \end{aligned}$$

we get

$$\frac{1}{n} \left| \sum_{k=1}^n [F(x_{k,n}(q)) - F(x_{k,n}(f))] \right| < \varepsilon \quad (3.6)$$

The approximations (3.5) and (3.6), with ε arbitrary, lead us, in the usual way, to deduce that (3.4) continues to hold when replaced by $f \in C(\text{supp}(d\alpha))$. This completes the proof of Theorem 1.

We remark, finally, that if in Theorem A we take the distribution function which yields the normalized Chebychev polynomials $\{T_n(x)\}_0^\infty$, then analysis similar to the above, but rather simpler, leads to the converse result that Theorem A implies Theorem B.

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