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INNER COMPOSITION OF ANALYTIC MAPPINGS ON THE UNIT DISK

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ABSTRACT. A basic theorem of iteration theory (Henrici [6]) states that f analytic on the interior of the closed unit disk D and continuous on D with Int(D) f(D) carries any point $z \varepsilon D$ to the unique fixed point $\alpha \varepsilon D$ of f. That is to say, $f^n(z) \to \alpha$ as $n \to \infty$. In [3] and [5] the author generalized this result in the following way: Let $F_n(z): = f_1 o \ldots o f_n(z)$. Then $f_n \to f$ uniformly on D implies $F_n(z) \to \lambda$, a constant, for all $z \varepsilon D$. This kind of compositional structure is a generalization of a limit periodic continued fraction. This paper focuses on the convergence behavior of more general inner compositional structures $f_1 o \ldots o f_n(z)$ where the f_j's are analytic on Int(D) and continuous on D with Int(D) $f_j(D)$, but essentially random. Applications include analytic functions defined by this process.

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1. INTRODUCTION.

Let f(z) be a function that is analytic on the interior of the unit closed disk $D = (|z| \leq 1)$ and continuous on D. Suppose that f(D) lies in the interior of D. It is well-known that f must have exactly one fixed point α in the set f(D) and the nth iterate $f^n(z)$ of any point $z \varepsilon D$ converges to α as $n \to \infty$. From Henrici (theorem 6.12a[6]) we have the following result with slightly more liberal hypotheses.

THEOREM 1. Let f be analytic in a simply connected region S and continuous on the closure S' of S, and let f(S') be a bounded set contained in S. Then f has exactly one fixed point and the sequence $\{f^n(z)\}$ converges to the fixed point for arbitrary $z \varepsilon S'$.

The proof of theorem 1 is predicated on the proof of the theorem for the special case in which S is the open unit disk. A simple application of the Riemann mapping may accelerate the convergence o these expansions [2], [3].

The following theorem due to Hillam and Thron (Lemma 4.38[7]) demonstrates the preliminary ideas under discussion in the context of fairly general Mobius transformations.

THEOREM 3. Suppose that $f_n(z) = (a_n z + b_n)/(c_n z + d_n)$, D $f_n(D)$, and $f_n(\infty) = k$, $|k| < 1$. Then $F_n(z) \to \lambda$ for all $z \in INT(D)$.

COROLLARY. The modified continued fraction $K(1/b_n)$ having nth convergent $1/b_1 + \dots + 1/(b_n+z) = f_1 o \dots of_n(z)$, where $f_n(z) = 1/(b_n + z)$ and $|b_n| > 2$, converges to a constant for $||z|| < 1$.

PROOF. The center C and radius r of $f_n(D)$ are $C = b_n/(|b_n|^2 - 1)$ and $r = C/b_n$, where $|b_n| > 1$. $||C|| + r < 1 \iff |b_n| > 2$. Therefore, D $f_n(D)|$

It is interesting to compare the convergence behaviors of "outer" and "inner" compositions. We shall see that the convergence of $\{F_n(z)\}$ is guaranteed if the f_n's map D into $(|w| < .6)$, e.g., whereas, it is trivial to find such functions that will produce oscillatory divergence of $\{J_n(z)\}$ no matter how small a disk $(|w| < R < 1)$ the f_n's map D into.

EXAMPLE 1. Let f(z) be a Mobius transformation mapping D onto $(|z - R/2| < R/8)$ and let g(z) be a similar function mapping D onto $(|z + R/21 < R/8)$. If $f_{2n}(z) = f(z)$ and $f_{2n-1}(z) = g(z)$, then $\{J_n(z)\}$ diverges for each $z \in D$.

2. CONVERGENCE THEOREMS.

We begin our expoloration of the convergence behavior of $\{F_n(z)\}$ with the observation that some kind of condition is required in order to insure convergence to a constant.

EXAMPLE 2. Let $f_n(z) = r_n e^{i\theta} z$ where $1 > r_n \to 1$, $\Pi r_n \neq 0$ and $\theta \neq 2\pi n$. Let $\Pi r_n = \rho > 0$ (e.g., $r_n = 1 - 1/n^2$). Then each f_n maps D into Int D, and $\{F_n(z_0)\}$ oscillates almost uniformly on the circle $(|z| = \rho|z_0|)$.

Next, we introduce a very simple lemma involving a Lipschitz condition on the f_n's. Set $F_{n,n+m}(z) := f_{n+1} o \dots o f_{n+m}(z)$.

LEMMA 1. (a). Let $U := (|z| < \rho < 1)$. Suppose that $z \in D \implies f_n(z) \in U$ for all n. If $|f_n'(z)| < K_n$ for all $z \in U$ and $\Pi K_n = 0$, then $F_n(z) \to \lambda$ for all $z \in D$.

PROOF. $|f_n(z_1) - f_n(z_2)| < K_n|z_1 - z_2|$ for $z_1, z_2 \in U$ implies

$|F_n(z) - F_{n+m}(z)| < (\Pi_1^{n-1} K_i)|f_n(z) - F_{n-1,n+m}(z)| < 2\rho(\Pi_1^{n-1} K_i)$

Hence $\{F_n(z)\}$ converges for each z in D. Furthermore,

$$|F_n(z_1) - F_n(z_2)| < 2\rho(\Pi_1 NK_i) \text{ implies } F_n(z) \to \lambda \text{ for each } z \in D$$

(or $F_n(D) \to \lambda$).|

LEMMA 1. (b) Let $U = (|z| < \rho < 1)$. Suppose that U $f_n(S,D)$ for $n > 0$. If theorem then suffices to extend the result to a more general set S.

The author, in [5], extended theorem 1 by considering limit periodic sequences of the form $\{F_n(z)\}$ where $F_1(z) = f_1(z)$, $F_n(z) = F_{n-1}(f_n(z))$, with $f_n \to f$ in a region S. (A slightly weaker result not requiring the Riemann mapping theorem is found in [3]).

THEOREM 2. Let f be analytic in a simply connected region S and continuous on the closure S' of S, and let f(S') be a bounded set contained in S. Suppose $f_n \to f$ uniformly on S. Then $F_n(z) \to \lambda$, a constant, for each $z \in S'$.

Limit periodic sequences occur naturally in the study of limit periodic continued fractions and quasi-geometric series, and may be generalized in complete metric spaces [2]. Such sequences when employed in the context of functional expansions are inherently more interesting and productively richer than simple iteration or what might be considered "outer" composition ($J_n(z) = f_n o f_{n-1} o \dots o f_1(z)$) for the following reason: When $f_n \to f$ and a simple Lipschitz condition holds on the f_n's these latter two sequences converge to the attractive fixed point of the limit function f, whereas the limit periodic sequence converges, but to a limit that depends upon the structures of the individual f_n's.

In the present paper the following question is posed, and, to some extent, answered: Suppose each member of the sequence $\{f_n\}$ is analytic on Int(D) and continuous on D with D $f_n(D)$ (it is not assumed that $f_n \to f$). Under what conditions does $F_n(z) = f_1 o \dots o f_n(z) \to \lambda$, a constant, for all $z \in D$, as $n \to \infty$? Thus we are considering "inner" compositions of essentially random sequences of functions mapping the unit disk into itself.

Although our approach focuses on mappings of D into D, more general results are possible. Let $S = \Gamma$ Int(Γ) where Γ is a Jordan curve. Let Φ be the Riemann mapping function giving $\Phi(S) = D$. Suppose that g_n is analytic in Int(Γ) and continuus on S, with $G_n(S)$ contained in S. Then $\Phi o g_n o \Phi^{-1} := f_n$ maps D into D. It easily follows that the convergence of $\{F_n\}$ implies the convergence of $\{G_n\}$ where $G_n(z) := g_1 o \dots o g_n(z)$.

We shall present several theorems describing conditions on the f_n's that imply $F_n(D) \to \lambda$. After proving each of these basic theorems we will exhibit an alternative and extended version of the result describing a class of analytic functions that can be generated in the following way: for each n let $f_n(z) = f_n(\zeta,z)$ be analytic for both $\zeta \in S$, a compact region, and $z \in D$. Let $F_n(\zeta,z) := F_{n-1}(\zeta,f_n(\zeta,z))$ with D $f_n(S,D)$. The fixed points of the f_n's are $\alpha_n = \alpha_n(\zeta)$ satisfying $f_n(\zeta,z) = z$. Then $F_n(\zeta,D) \to \lambda(\zeta)$ uniformly on S, and $\lambda(\zeta)$ is analytic on S.

Apart from elementary details concerning unform boundedness and uniform convergence, the proof of these alterntive theorems are pratically identical to the proofs that are given for the simpler versions, and are therefore omitted. This will minimize notational complexity.

Although the method of constructing $\lambda(\zeta)$ seems unusual several common modes of functional expansion may be categorized in this way. In fact, a judicious choice of z $|\partial f_n(\zeta,z)/\partial z| \leq K_n$ for all $z \in U$ and for all $\zeta \in S$, and $\Pi K_n = 0$, then $F_n(\zeta,D) \to \lambda(\zeta)$ uniformly on S.

We then easily obtain a result concerning the case in which the f_n's map D into a smaller circle whose center is the origin.

THEOREM 4. (a) Suppose $|f_n(z)| \leq R := (\sqrt{5} - 1)/2 < .6181$ for all n for $|z| \leq 1$. Then $F_n(D) \to \lambda$.

PROOF. Set $g_n(z) = f_n(z)/R$. Then $|g_n(z)| \leq 1$ for $|z| < 1$ implies $|g_n'(z)| \leq (1-|g_n(z)|^2)/(1-|z|^2) \leq 1/(1-R^2)$ if $|z| \leq R$. This result follows from Schwarz's lemma and may be found, e.g., in [9]. Therefore, in $F_n(z)$, $|f_k'(z)| \leq K = R/(1-R^2) < 1$ for $k < n$, and lemma 1 applies.

THEOREM 4(b). Suppose for all n $|f_n(\zeta,z)| \leq R < (\sqrt{5} - 1)/2 < .6181$ for all $\zeta \in S$ and for all $z \in D$. Then $F_n(\zeta,D) \to \lambda(\zeta)$ uniformly on S.

The fact that $|f_n(z)| \leq R < 1$ for all $|z| \leq 1$ is not sufficient to guarantee the Lipschitz condition $|f_n'(z)| < 1$ for $|z| \leq R$. This can be easily seen in the example

$f_n(z) = .9\, z^5$. Here $|f_n(z)| \le .9$ for $|z| \le 1$, and $|f_n'(.9)| > 2.95$.

EXAMPLE 3. Set $f_n(\zeta,z) := e^{\zeta a+\omega(n)}$ for $\mathrm{Re}(\omega(n)) < 1.5$, $|\zeta| \le 1$, $|z| \le 1$. Then $|f_n(\zeta,z)| \le R < .61$ for the indicated values of ζ and z. Therefore $F_n(\zeta,z) \to \lambda(\zeta)$, analytic on $(|\zeta| \le 1)$.

EXAMPLE 4. We define a continued square fraction by setting $f_n(\zeta,z) := a_n(\zeta)/(b_n(\zeta) + z^2)$ for $\zeta \in S$ and $z \in D$. If we assume that $|b_n(\zeta)| \ge 2$ and $|a_n(\zeta)| \le R < (\sqrt{5}-1)/2$ for $\zeta \in S$, then $|f_n(\zeta,z)| \le R$ and $F_n(\zeta,z)$ $\lambda(\zeta)$, analytic on S.

EXAMPLE 5. Sometimes a power series $P(\zeta) := a_1\zeta + a_2\zeta^2 + \ldots$ may be formally converted into an expansion having the form $F_n(\zeta,z)$ where $f_n(\zeta,z) := \zeta/\sqrt{(b_n+z)}$. If $|b_n| \ge 2$ when $|\zeta| \le R < 1$ and $|z| \le 1$, then $F_n(\zeta,z) \to \lambda(\zeta)$, analytic on $(|\zeta| \le 1)$.

If the values of $f_n(0)$ are fairly close to 0, the critical value of R can be a bit larger.

THEOREM 5. (a) Let R_0 be the (positive) root of $P(x) = x^4 + x - 1$. $(R_0 \simeq .7244)$. If $|f_n(z)| \le R < R_0$ for all $z \in D$ and $|f_n(0)| \le \varepsilon < \mathrm{Min}\ \{R-R^2, (\sqrt{(1-R)} - R^2)\}$ for all n, then $F_n(D) \to \lambda$.

PROOF. Consider $|f_n(z)| \le R$ for $|z| \le 1$. Let $H_n(z) = f_n(z)/R$. Then $|H_n(z)| \le 1$ for $|z| \le 1$. Set $a_n = f_n(0)/R$ $(|a_n| < 1)$. The proof involves a more or less routine extension of Schwarz's lemma that begins with the observation that the linear fractional transformation $t_n(z) = (z - a_n)/(1 - \bar{a}_n z)$ maps the unit disk onto itself, and, consequently, $|w| \le 1$ if $|z| \le 1$ where $w = T_n(z) = (H_n(z) - a_n)/(1 - a_n H_n(z))$. Since $T_n(0) = 0$, we have $|T_n(z) \le |z|$ if $|z| \le 1$.

Solving for $H_n(z)$, $H_n(z) = (a_n + T_n(z))/(1 + \bar{a}_n/T_n(z))$, so that

$|H_n(z)| \le ((|a_n| + |T_n(z)|)/(1 + |a_n||T_n(z)|) \le (|a_n| + |z|)/(1 + |a_n||z|)$ for $|z| \le 1$.

Hence, if $|z| \le R$, we shall have
$|f_n(z)| \le R(R + |a_n|)/(1 + |a_n|R) \le (R^2 + \varepsilon)/(1 + \varepsilon)$. Thus $|f_n(z)| < R^2 + \varepsilon < R$ if $|z| \le R$.

Using the standard estimate for the derivative occuring in the proof of theorem 3, $|H_n'(z)| \le (1 - |H_n(z)|^2)/(1 - |z|^2)$ for $|z| < 1$.

Restricting $|z| < R^2 + \varepsilon < 1$, this leads immediately to $|f_n'(z)| \le R/(1 - (R^2 + \varepsilon)^2)$ which is less than one if $\varepsilon < \sqrt{(1-R)} - R^2$. This last expression is greater than zero if $R < R_0$.

Next, we write $F_n(z) = f_1 o \ldots o f_{n-2} o f_{n-1} o f_n(z) = f_1 o \ldots o f_{n-2}(z_{n-1})$ where $z_n = f_n(z)$ and $z_{n-1} = f_{n-1}(z_n)$. Then $|z| \le 1 \implies |z_n| \le R \implies |z_{n-1}| < R^2 + \varepsilon$. Consequently, lemma 1 applies.

THEOREM 5(b). Let R_0 be the (positive) root of $P(x) = x^4 + x - 1$. $(R_0 \simeq .7244)$. If $|f_n(\zeta,z)| \le R < R_0$ for all $\zeta \in S$ and for all $z \in D$, and

$|f_n(\zeta,0)| \le \varepsilon < (\min\ \{R - R^2, \sqrt{1 - R} - R^2\}$ for all $\zeta \in S$, then $F_n(\zeta,D) \to \lambda(\zeta)$ uniformly on S.

We turn now to conditions on the fixed points of the f_n's that insure convergence of $\{F_n(\zeta,z)\}$. Let $f_n(\zeta,z) = z \Longleftrightarrow z = \alpha_n(\zeta)$. Investigations of limit periodic phenomena suggest that these fixed points may play a strong role in the kind of generalized iteration now being explored [1], [3], [4], [8].

Our next theorem is, in a sense, a generalization of theorem 1.

THEOREM 6(a). Suppose that $|f_n(z)| \leq R < 1$ for all n for all $z \in D$, and that $\alpha_n \to \alpha$. Then $F_n(D) \to \lambda$.

PROOF. Set $T(z) = (z - \alpha)/(1 - \alpha z)$. Then $T(D) = D$ and $T(\alpha) = 0$. Let $g_n(z) = T o f_n o T^{-1}(z)$. Then $|g_n(z)| \leq r = (R + |\alpha|)/(1 + |\alpha|R) < 1$ if $|z| < 1$, since $\operatorname{Max}_{|z| \leq R}|T(z)| = (R + |\alpha|)/(1 + |\alpha|R) = r < 1$ if $R < 1$.

Set $a_n = g_n(0)/r$. Then $a_n \to 0$ as $n \to \infty$. (This follows from the fact that $f_n(\alpha) \to \alpha$, since $|f_n(\alpha) - \alpha_n| = |f_n(\alpha) - f_n(\alpha_n)| \leq \operatorname{Max}_{|w| \leq R}|f_n'(w)||\alpha - \alpha_n|$). Thus $|g_n(z)/r| \leq 1$ if $|z| \leq 1$ and $g_n(0)/R = a_n$.

Now, using the extension of Schwartz's lemma occuring in the proof of thoerem 5,

(1) $|g_n(z)/r| \leq (|a_n| + |z|)/(1 + |a_n||z|) \Longrightarrow |g_n(z)| \leq r|a_n| + r|z|$ if $|z| \leq 1$.

Therefore,

$$|g_n g_{n+1} o \ldots o g_{n+m}(z)| \leq r|a_n| + r^2|a_{n+1}| + \ldots + r^{m+1}|a_{n+m}| + r^{m+1}|z|$$

$$\leq r\varepsilon + r^2\varepsilon + \ldots + r^{m+1}\varepsilon + r^{m+1} \text{ if n is sufficiently large.}$$

$$\leq r\varepsilon/(1-r) + r^{m+1}.$$

Recall that $g_{n,n+m}(z) = g_n o \ldots o g_{n+m}(z)$.

Thus

(2) For $\delta > 0$ there exist n_0, m_0 such that $n \geq n_0$ and $m \geq m_0$ imply

$$|G_{n,n+m}(z)| < \delta.$$

Now, (1) implies $|z| \leq \varepsilon \Longrightarrow |g_n(z)| \leq \varepsilon$ if ε is sufficiently small and n is sufficiently large.

Therefore, for large n and m, $|G_{n,n+m}(z)| \leq \varepsilon$ and $|g_k(G_{n,n+m}(z)| \leq \varepsilon$ provided k is large.

We will now show, in three steps, that $|f_n'(\alpha_n)| \leq \rho < 1$ for all n and that this implies $|g_n'(z)| \leq K < 1$ for all n sufficiently large and for all $|z|$ sufficiently small. It will then be possible to use this information to establish the convergence of $\{G_{n,n+m}(z)\}$as $m \to \infty$.

1. For each N set $t_n(z) = (z - \alpha_n)/(1 - \alpha_n z)$ and $h_n(z) = t_n o f_n o t_n^{-1}(z)$ where $t_n(\alpha_n) = 0$. Thus $h_n(0) = 0$.

Since $|t_n(w)| \leq (|\alpha_n|+R)/(1+|\alpha_n|R) \leq (a+R)/(1+aR) = \rho < 1$ where $|\alpha_n| \leq a \leq R$,

$|h_n(z)| \leq \rho < 1$ for all n and all $|z| \leq 1$. Thus

$|t_n'(\alpha_n)||f_n'(\alpha_n)||t_n^{-1}{}'(0)| = |h_n'(0)| \leq \rho < 1 \Longrightarrow |f_n'(\alpha_n)| \leq \rho < 1$ for all n.

2. There exists $\varepsilon_1 > 0$ such that $|z - \alpha| < \varepsilon_1 \Longrightarrow |f_n'(z)| \leq \rho_1 < 1$ for n suffIcently large. If this were not the case there would exist $\{z_n\}$ such that $z_n \to \alpha$ and $|f_n'(z_n)| \geq 1 - 1/n + 1$. However

$$\left|f_n'(\alpha_n) - f_n'(z_n)\right| \leq \mathrm{Max}_{|w|<R}\left|f_n''(w)\right| \left|\alpha_n - z_n\right| \to 0 \ (\to\infty).$$

3. $\left|g_n'(z)\right| = \left|T'(F_n(T^{-1}(z)))\right| \left|f_n'(T^{-1}(z)))\right| \left|T^{-1}(z)\right| \leq K < 1$ for all $|z| \leq \varepsilon \leq \varepsilon_1$ and for large n. (Even though $\{f_n\}$ does not converge, $T^{-1}(z)$ close to α implies $f_n(T^{-1}(z))$ is uniformly close to α since $\left|f_n(w) - f_n(\alpha)\right| \leq \rho_1|w - \alpha|$ and $f_n(\alpha) \to \alpha$.)

We are now ready to show that $\mathrm{Lim}_{n\to\infty} G_{k,k+n}(z) = C$ for all $|z| \leq 1$.

Fix k such that $j \geq k$ and $|z| \leq \varepsilon \Longrightarrow |g_j(z)| \leq \varepsilon$. Fix $m = m_0$. Let $n \geq n_0$.

Set $\Gamma_n = g_k o \ldots o g_{k+n} o \ldots o g_{k+m+n}(z)$.

Then $\left|\Gamma_n - \Gamma_{n+p}\right| \leq K^{n+1}(2\varepsilon) \to 0$ as $n \to \infty$.

Hence $\mathrm{Lim}_{n\to\infty} G_{k,k+m+n}(z) = C_k(z)$ exists, and it is easily shown that $C_k(z) = C_k$ for all $|z| \leq 1$. Therefore

$\mathrm{Lim}_{n\to\infty} g_1 o \ldots o g_{k-1}(G_{k,k+m+n}(z)) = g_1 o \ldots o g_{k-1}(C_k) = C$. It then follows that

$F_n(z) = T^{-1} o\ g_1\ o \ldots o g_n o\ T(z) \to T^{-1}(C) = \lambda.$ |

Comment: If $\alpha_n \equiv \alpha$, then $\lambda = \alpha$.

THEOREM 6(b). Suppose for all n $\left|f_n(S,D)\right| \leq R < 1$ and $\alpha_n(\zeta) \to \alpha(\zeta)$ uniformly on S. Then $F_n(\zeta,D) \to \lambda(\zeta)$ uniformly on S.

EXAMPLE 6. We obtain an α-limit periodic [2] (i.e., $\{\alpha_n\}$ converges, but $\{f_n\}$ does not) continued square fraction by setting

$f_n(\zeta,z) := r_n\ \alpha_n(\zeta)/(r_n - \alpha_n(\zeta)^2 + z^2)$ where $\left|\alpha_n(\zeta)\right| < \rho$ for $\zeta \in S$, $\alpha_n(\zeta) \to \alpha(\zeta)$ uniformly on S, and $2 + \rho^2 \leq |r_n| \leq R/\rho$ for $R < 1$. These conditions insure that $\left|f_n(\zeta,z)\right| \leq R < 1$ for $|z| \leq 1$. Thus $F_n(\zeta,z) \to \lambda(\zeta)$, analytic on S.

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