A NOTE ON THE CONVOLUTION AND THE PRODUCT D' AND S'

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ABSTRACT. Examples of tempered distributions are shown such that the convolution and product exist in D' and are tempered distributions, but they do not exist in S'. On the other hand, there exist tempered distributions whose convolution or product exist in D', but are not tempered distributions.

KEY WORDS AND PHRASES. Tempered Distributions, Convolution and Product of Distributions.

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1. INTRODUCTION.

Various definitions of the convolution and the product have been considered in the space D' of distributions and in the space S' of tempered distributions, the duals to the spaces D and S, respectively (see Schwartz [1], pp. 21, 24, 233, 237).

In the case of the convolution f*g in D' (and S' resp.), most of the known definitions, e.g. those given by Schwartz [1] (p. 153), Chevalley [2], Shiraishi [3], Vladimirov [4] (p. 62), Dierolf and Voigt [5], are equivalent. For our purposes, we will consider the convolution in D' and in S' defined by means of so-called unit-sequences, which are equivalent to the mentioned definitions for the respective classes of unit-sequences as shown in Kamiński [6]. By changing classes of unit-sequences one can obtain, however more general definitions of the convolution in D' and in S'.

The definitions of the product f.g in D' (in S' resp.) considered in the literature are not equivalent. For instance, the definition for D' given by Hirata and Ogata [7] and by Mikusiński [8] by means of so-called delta-sequences depends essentially on the class of delta-sequences taken into consideration. Notice that given two tempered distributions it makes sense to consider the two operations of the convolution and product: in D' and in S'. Therefore the following questions appear:

(I) Let f, g ε S'. Suppose that f*g (f.g, resp.) exists in D'. Does f*g (f.g, resp.) belong to S'?

(II) Let f, g ε S'. Suppose that f*g, (f.g, resp.) exists in D' and belongs to S'. Does f*g (f.g, resp.) exist in S'?

The first question in the case of the convolution was posed by Shiraishi [3] and answered negatively by Kamiński [9] (see also Kamiński [10]) and by Dierolf and Voigt [5] (see also Ortner and Wagner [11]). For the case of the product see Section 3.

The answer to the second question is also negative. A respective counter-example for the convolution in R^2 was given by Wagner [12]. He also showed that the existence of the convolution f*g in D' of homogeneous tempered distributions f, g implies the existence of f*g in D'. Since Wagner's counter-example cannot be transferred into the one-dimensional case, we give a suitable example in Section 4. This counter-example can be automatically extended to R^d. Let us underline that in our example f = g and f is a real-valued distribution.

In the case of the product, a respective counter-example could be obtained from the example constructed for the convolution by means of the Fourier transform. This requires, however, careful considerations on relations between various definitions of the product of tempered distributions. Let us notice that the tempered distribution obtained in this way is complex-valued. Therefore we give in Section 5 a straightforward example with real-valued tempered distributions.

2. DEFINITIONS.

By E we denote the class of all unit-sequences, i.e. sequences (η_n) of functions of the class D satisfying the conditions:

(i) $D^k(\eta_n - 1) \to 0$ almost uniformly on R^d for every multi-index $k = (\kappa_1, \ldots, \kappa_d) \in N_o^d$, where $N_o = N \cup \{0\}$.

(ii) For every $k \in N_o^d$ there is a constant $M_k > 0$ such that

$$\left| D^k \eta_n(x) \right| < M_k$$

for all $x = (\xi_1, \ldots, \xi_d) \in R^d$ and $n \in N$.

By E_m we denote the class of all model unit-sequences, i.e. sequences (η_n) of the form

$$\eta_n(x) = \eta(\alpha_n x), \quad x \in R^d, \ n \in N,$$

where η is a function from D such that $\eta(0) = 0$ and (α_n) is a sequence of real numbers converging to 0. Evidently, we have $E_m \subset E$.

Let f, g $\in D'$ (f, g $\in S'$, resp.). We say that the convolution f*g exists in D' (in S', resp.) if the limit

$$\lim_{n \to \infty} (\eta_n f) * (\tilde{\eta}_n g) \tag{2.1}$$

exists in D' (in S', resp.) for all (η_n), $(\tilde{\eta}_n) \in E$. If

$$\lim_{n \to \infty} (\eta_n f) * (\eta_n g) \tag{2.2}$$

exists for all $(\eta_n) \in E$, we say that the symmetric convolution f*g exists in D' (in S'). In both cases the limits do not depend on unit-sequences of the class E. In

the above definitions the class E can be replaced by E_m, but then we additionally assume that the limits (2.1) and (2.2) do not depend on model unit-sequences and f*g is called the model (symmetric) convolution in $\mathcal{D}'$ (in S').

As shown in [6], the convolution in $\mathcal{D}'$ (in S') as above is equivalent to the convolution in $\mathcal{D}'$ (in S') in the sense of the other known definitions mentioned in Section 1.

If the convolution f*g exists in $\mathcal{D}'$ (in S'), then the symmetric convolution f*g exists in $\mathcal{D}'$ (in S'), but not conversely. If the (symmetric) convolution f*g exists in $\mathcal{D}'$ (in S'), then the model (symmetric) convolution f*g exists in $\mathcal{D}'$ (in S'), but not conversely.

By Δ_o we denote the class of all delta-sequences (δ_n), i.e. sequences of smooth functions satisfying the conditions:

(iii) there exists a sequence of positive numbers α_n, $\alpha_n \to 0$ such that

$$\delta_n(x) = 0 \text{ for } |x| = (\xi_1^2 + \ldots + \xi_d^2)^{1/2} \geqslant \alpha_n \text{ and } n \in N;$$

(iv) $\int \delta_n(t) \, dt = 1$ for $n \in N$;

(v) there exists an $M > 0$ such that

$$\int |\delta_n(t)| < M \text{ for } n \in N.$$

By Δ_m we denote the class of all model delta-sequences of the form

$$\delta_n(x) = \alpha_n^{-d} \sigma(\alpha_n^{-1} x), \qquad x \in R^d, \, n \in N,$$

where $\sigma \in \mathcal{D}$, $\int \sigma = 1$ and (α_n) is a sequence of real numbers converging to 0. Evidently, $\Delta_m \subset \Delta_o$.

Let f, g $\in \mathcal{D}'$ (f, g $\in S'$, resp.). We say that the product f·g exists in $\mathcal{D}'$ (in S', resp.) if the limit

$$\lim_{n \to \infty} (f * \delta_n) \cdot (g * \tilde{\delta}_n) \qquad (2.3)$$

exists in $\mathcal{D}'$ (in S', resp.) for all (δ_n), $(\tilde{\delta}_n) \in \Delta_o$. If

$$\lim_{n \to \infty} (f * \delta_n) \cdot (g * \delta_n) \qquad (2.4)$$

exists for all $(\delta_n) \in \Delta_o$, we say that the symmetric product exists in $\mathcal{D}'$ (in S'). If Δ_o is replaced by Δ_m, f·g is called the model (symmetric) product in $\mathcal{D}'$ (in S'). One can show that the limits (2.3) and (2.4) do not depend on delta-sequences from Δ_o. We assume this for the class Δ_m.

For these and other definitions of the product in $\mathcal{D}'$ and in S' and their equivalence see Mikusiński [8] and [13], Hirata and Ogata [7], Shiraishi and Itano [14], Antosik, Mikusiński and Sikorski [15] (p. 242), Kamiński [6], Oberguggenberger [16], and Wawak [17, 18]. Note that if the product f·g exists in $\mathcal{D}'$ (in S'), then the symmetric product f·g exists in $\mathcal{D}'$ (in S'), but not conversely. If the

(symmetric) product f.g exists in $\mathcal{D}$ (in S'), then the model (symmetric) product f.g exists in $\mathcal{D}'$ (in S'), but not conversely.

3. QUESTION (I) FOR PRODUCT.

It is very easy to give the negative answer to the question (I) for the product in R^1 in case complex-valued distributions (functions) are admitted. Namely, given an arbitrary real-valued continuous function F on R^1 (in particular, F can be chosen so that F $\notin S'$) put

$$G(x) = \exp \left[i \int_0^x F(t)dt \right],$$

$$f_1(x) = G'(x) = iG(x)F(x)$$

and

$$f_2(x) = -iG^{-1}(x)$$

for $x \in R^1$. Then f_1, f_2 are continuous complex-valued functions on R^1 such that $f_1 \cdot f_2 = F$ in the usual sense, so the more in $\mathcal{D}'$, and f_1, $f_2 \in S'$, because $|G(x)| = 1$.

We are going to show that there exists also a real-valued smooth function ϕ which belongs to S' such that the square ϕ^2 (which also exists in the sense of the product in $\mathcal{D}'$) does not belong to S'. First denote $I_j = [j-3^{-j}, j+3^{-j}]$, $J_j = [j-2.3^{-j}, j+2.3^{-j}]$ for $j \in N$. Let ϕ be a smooth function such that

$$\phi(x) = 2^j, \qquad\qquad x \in I_j, \qquad\qquad (3.1a)$$

$$0 < \phi(x) < 2^j, \qquad\qquad x \in J_i, \qquad\qquad (3.1b)$$

and

$$\phi(x) = 0, \qquad\qquad x \notin \bigcup_{j=1}^{\infty} J_j \qquad\qquad (3.1c)$$

for $j \in N$. By (3.1) we have

$$\int_0^\infty \phi(t)\, dt < \sum_{j=1}^{\infty} 2^j |J_j| = 4 \sum_{j=1}^{\infty} (2/3)^j < \infty,$$

so $\phi \in S'$. Since $\phi * \delta_n \to \phi$ almost uniformly for any $(\delta_n) \in \Delta_0$, the product $\phi \cdot \phi$ exists in $\mathcal{D}'$ and coincides with the usual product ϕ^2 of functions. Taking $\omega(x) = \exp(-x/4)$, we have

$$\int_0^\infty \phi^2(x)\omega(x)\, dx > 2\alpha \sum_{j=1}^{\infty} (4\alpha/3)^j = \infty,$$

where $\alpha = \exp(-1/4)$. Consequently, $\phi^2 \in S'$.

Another example, given explicitly by $\phi(x) = e^x \cos e^x$ but based on similar calculations, was communicated to us by P. Wagner.

4. QUESTION (II) FOR CONVOLUTION.

Now, we shall construct an example of a distribution $f \in S'$ such that the convolution square f*f exist in $\mathcal{D}'$ and is of the class S', but does not exist in S'.

Let $f = f_1 + f_2$ with

$$f_1(x) = \sum_{j=2}^{\infty} (-1)^j \, 2^{[\log_2 j]} \, \delta(x - 2^j)$$

and

$$f_2(x) = \sum_{j=2}^{\infty} \delta(x + 2^j - [\log_2 j]),$$

where the symbol $[x]$ means the integer part of x. It is clear that f_1, f_2, $f \in S'$. Note that

$$A = \text{supp } f_1 = \{2^k : k \in N \setminus \{1\}\},$$

$$B = \text{supp } f_2 = \{-2^k + [\log_2 k] : k \in N \setminus \{1\}\}.$$

Each of the sets $\text{supp } f_1$ and $\text{supp } f_2$ is bounded from one side, so it is polynomially compatible in the sense of Kamiński [10]. Therefore the convolutions $f_1 * f_1$ and $f_2 * f_2$ exists in S' (cf. [9] and [10]). It suffices to show that the convolution $f_1 * f_2 = f_2 * f_1$ exists in $\mathcal{V}'$ and does not exist in S'. To prove that $f_1 * f_2$ exists in $\mathcal{V}'$ it is enough to verify that for every bounded interval I in R^1 the set

$$\{(x,y): x \in A, \; y \in B, \; x + y \in I\}$$

is bounded in R^2. But if

$$\left| 2^j - 2^k + [\log_2 k] \right| < m$$

for some $m \in N$, then we have

$$\max(j,k) < 2 + \log_2 m \qquad \text{for } j \neq k$$

and

$$j, k < 2^{m+1} \qquad \text{for } j = k,$$

which proves the required property. In addition, we have

$$f_1 * f_2 = \sum_{j=2}^{\infty} \sum_{k=2}^{\infty} (-1)^j \, 2^{[\log_2 j]} \, \delta(x - 2^j + 2^k - [\log_2 k]).$$

In order to prove that $f_1 * f_2 \in S'$, fix $m \in Z$ and consider the pairs (j,k) for which

$$2^j - 2^k + [\log_2 k] = m. \qquad\qquad (4.1)$$

If $j = k$ then

$$j = k \in \{2^m, \, 2^m + 1, \, \ldots, \, 2^{m+1} - 1\}.$$

If $j \neq k$ then there exists at most one pair (j,k) such that (4.1) holds and $|m| > 2^{j-1} > j$. Setting $J = \{j \in N: 2^m < j < 2^{m+1} - 1\}$, we obtain

$$\sum_{j \in J} (-1)^j \; 2^m \delta(x-m) = 0. \qquad\qquad (4.2)$$

Hence

$$(f_1 * f_2)(x) = \sum_{m \in Z} a_m \delta(x-m),$$

where $a_m = (-1)^j \; 2^{[\log_2 j]}$ if there exists a pair (j,k) such that (4.1) holds and $a_m = 0$ otherwise.   In both cases $|a_m| \le j \le m$, which implies that $f_1 * f_2 \in S'$.   Now, let η be a real-valued function of the class $\mathcal{D}$ such that $\eta(x) = 1$ for $|x| \le 1$ and $\eta(x) = 0$ for $|x| \ge 3/2$.   Setting $\eta_n(x) = \eta(x/n)$ we have, for $n = 2^{2^m}$,

$$\eta_n(x) f_1(x) = \sum_{j=2}^{2^m} (-1)^j \; 2^{[\log_2 j]} \delta(x-2^j)$$

and

$$\eta_n(x) f_2(x) = \sum_{k=2}^{2^m} \delta(x+2^k - [\log_2 k]).$$

From the last formulae it follows that the distribution $(\eta_n f_1) * (\eta_n f_2)$ can be expressed as a sum of two distributions $g_m = \Sigma_1 \phi_{jj}$ and $h_m = \Sigma_2 \phi_{jk}$, where

$$\phi_{jk} = (-1)^j \; 2^{[\log_2 j]} \delta(x - 2^j + 2^k - [\log_2 k])$$

and Σ_1 denotes the summation over all j with $2 \le j \le 2^m$ and Σ_2 denotes the summation over all j and k with $2 \le j, k \le 2^m$, $j \ne k$.   Setting $h = \Sigma_3 \phi_{jk}$, where Σ_3 denotes the summation over all $j,k \ge 2$ with $j \ne k$, it is easy to check that

$$\lim_{m \to \infty} \langle h_m, \phi \rangle = \langle h, \phi \rangle$$

for all $\phi \in S$.   On the other hand, taking $\phi \in S$ such that $\phi(x) = (2/3)^x$ for $x > 0$, we have, in view of (4.2),

$$\langle g_m, \phi \rangle = \langle 2^m \delta(x-k), \phi(x) \rangle$$

$$= 2^m \phi(k) = (4/3)^m \to \infty$$

as $m \to \infty$.   Therefore the convolutions $f_1 * f_2 = f_2 * f_1$ do not exist in S' and this means that $f*f$ does not exist in S'.

    REMARK 4.1.   Note that in the construction above $f*f$ does not exist in S' even in the sense of the symmetric model convolution.   Another example of a distribution $f \in S'$ such that the convolution $f*f$ exists in $\mathcal{D}'$ and does not exist in S' can be obtained by taking $f = g_1 + g_2$ with

$$g_1(x) = \sum_{k=2}^{\infty} (-1)^k \delta(x - 2^k), \quad g_2(x) = \sum_{k=2}^{\infty} \delta(x+2^k - [\log_2 k]).$$

In this example $f*f$ exists in S' in the sense of the limit (2.3) for all unit-sequences (η_n), $(\tilde{\eta}_n)$ with a commonly bounded variation.   In paricular, the model convolution $f * f$ exists in S', in contrast to the example presented before.

REMARK 4.2. Put $F = f * \omega$, where f is the distribution constructed in the example above and ω is a function of the class $\mathcal{D}$. Then F is a slowly increasing (i.e. bounded by a polynomial) smooth function such that the convolution $F*F$ exists in $\mathcal{D}'$ and represents a slowly increasing smooth function while the (model symmetric) convolution $F*F$ does not exist in S' . The example can also be modified in such a way that f is a distribution (even a smooth function) from an arbitrary Gelfand-Shilov space $K'(M_p)$ with the property: the convolution $f*f$ exists in $\mathcal{D}'$ and is a distribution (smooth function) of the class $K'(M_p)$, but $f*f$ does not exist in $K'(M_p)$.

5. QUESTION (II) FOR PRODUCT.

Now we are going to give a simple example of two tempered distributions f,g on R^1 such that the product $f.g$ exists in $\mathcal{D}'$ and belongs to S', but $f.g$ does not exist in S'. Let

$$f(x) = \sum_{k=1}^{\infty} \delta(x-k), \qquad g(x) = \sum_{k=1}^{\infty} \delta(x-k-2^{-k}).$$

Obviously, $f,g \in S'$. Since the product $\delta(x - a) \delta(x - b)$ exists in $\mathcal{D}'$ and equals 0 for $a \neq b$, it follows that the product $f.g$ exists in $\mathcal{D}'$ and $f.g = 0$. To prove that the symmetric model product $f.g$ does not exist in $\mathcal{D}'$ define

$$\delta_n(x) = n\omega(nx), \qquad (5.1)$$

where ω is a non-negative function of the class $\mathcal{D}$ such that

$$\omega(x) = 1 \quad \text{for } x \in [-1/4, 1/4] \qquad (5.2)$$

and

$$\int \omega(x)\,dx = 1. \qquad (5.3)$$

Moreover, consider a positive function ϕ of the class S such that $\phi(x) = (2/3)^x$ for $x > 0$. Then we have

$$\langle (f * \delta_n) \cdot (g * \delta_n), \phi \rangle > \sum_{m=1}^{\infty} \int \delta_n(x-m)\delta_n(x - m - 2^{-m})\,\phi(x)\,dx$$

$$> \int \delta_n(x - j)\,\delta_n(x - j - 2^{-j})\,\phi(x)\,dx > 2^{j-4}(2/3)^{j+1},$$

where $j = n + 2$. Since $2^{j-4}(2/3)^{j+1} \to \infty$ as $j \to \infty$, the symmetric model product $f.g$ does not exist in S'.

Finally, let us pose the following open question:

PROBLEM. Does there exist a real-valued tempered distribution f such that the product $f.g$ exists in $\mathcal{D}'$ and belongs to S', but the (symmetric model) product $f.f$ does not exist in S'?

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