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SUBMANIFOLDS OF EUCLIDEAN SPACE WITH PARALLEL MEAN CURVATURE VECTOR

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ABSTRACT. The object of the paper is to study some compact submaniforlds in the Euclidean space R^n whose mean curvature vector is parallel in the normal bundle. First we prove that there does not exist an n-dimensional compact simply connected totally real submanifold in R^{2n} whose mean curvature vector is parallel. Then we show that the n-dimensional compact totally real submanifolds of constant curvature and parallel mean curvature in R^{2n} are flat. Finally we show that compact Positively curved submanifolds in R^n with parallel mean curvature vector are homology spheres. The last result in particular for even dimensional submanifolds implies that their Euler poincare' characteristic class is positive, which for the class of compact positively curved submanifolds admiting isometric immersion with parallel mean curvature vector in R^n, answers the problem of Chern and Hopf.

KEY WORDS AND PHRASES. Submanifolds, totally real submanifolds, homology sphere, Euler-poincare' characteristic.

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1. Let g be the flat metric of R^n, $\overline{\nabla}$ be the corresponding Riemannian connection. If M is a submanifold of R^n with normal bundle ν, then the connection $\overline{\nabla}$ induces the Riemannian connection

∇ on M and the connection $\overset{\perp}{\nabla}$ in the normal bundle ν, and we have

$$\bar{\nabla}_X Y=\nabla_X Y+h(X,Y),\quad \bar{\nabla}_X N=-A_N X+\overset{\perp}{\nabla}_X N,\quad X,Y\in\chi(M),\ N\in\nu \tag{1.1}$$

where the second fundmental form $h(X,Y)$ is related to $A_N X$, by $g(h(X,Y),N) = g(A_N X,Y)$ and $\chi(M)$ is the Lie-algebra of vector fields on M. The mean curvature vector H of M is given by $H = 1/n\ \Sigma_{i=1}^{n} h(e_i,e_i)$, where $\{e_1,e_2,\dots,e_n\}$ is a local orthonormal frame of M. The mean curvature vector H is said to be parallel if $\nabla_X H = 0$, $X\in\chi(M)$. If $H = 0$ at each point of M, then M is said to be a minimal submanifold. It is known that if M is a compact submanifold of R^n, then M is not a minmal submanifold (cf.[1]).

The even dimensional Euclidean space R^{2n} has complex structure J which is parallel with respect to the connection $\bar{\nabla}$ that is, R^{2n} is a kaehler manifold. A submanifold M of R^{2n} is said to be totally real if $JTM \subseteq \nu$, where TM is the tangent bundle of M. In the case dim M = n and M is totally real submanifold of R^{2n}, using (1.1), we obtain

$$\overset{\perp}{\nabla}_X JY = J\nabla_X Y \text{ and } h(X,Y) = JA_{JY}X,\ X,Y\in\chi(M). \tag{1.2}$$

2. In this section we study the n-dimensional totally real submanifold M of R^{2n} with parallel mean curvature vector H, first under the topological restriction on M that, it is compact and its first Betti number is zero, and then under the geometric restriction that it is a space of constant curvature.

THEOREM 2.1. There does not exist an n-dimensional compact totally real submanifold with first Betti number equal to zero and with parallel mean curvature vector in R^{2n}.

PROOF. Let M be an n-dimensional compact totally real submanifold of R^{2n} with parallel mean curvature vector H. Then JH is a parallel vector field on M.

The 1-form η dual to JH is also parallel and hence harmonic. If $H^1(M;\mathbb{R}) = 0$, then η and hence H must vanish. But this would mean that M is a compact minimal submanifold of R^{2n}, which is impossible (cf. [1]).

THEOREM 2.2. Let M be an n-dimensional ($n \geq 2$) compact totally real submanifold of constant curvature in R^{2n} with parallel mean curvature vector. Then M is flat.

PROOF. If the curvature is nonzero constant, then M is irreducible and cannot admit a nonzero parallel vector field JH.

3. In this section we shall be concerned with the positively curved submanifolds with parallel mean curvature vector in R^n. We prove the following.

THEOREM 3.1. Let M be a compact and connected positively curved submanifold with parallel mean curvature vector in R^n. Then M is a homology sphere.

PROOF. Sicne M is compact, connected and $\overset{\perp}{\nabla}_X H = 0$, the function $\alpha = \|H\|$ is a non-zero constant. Define the unit normal vector field N on M by $N = 1/_{\alpha} H$. If $\Phi : M \to R^n$ is the immersion of M as submanifold of R^n, then the height function f_N: $M \to R$ is defined by $f_N(p) = g(\Phi(p), N)$. The hessian of the height function at a critical point $p \in M$ of f_N is given by the weingarten map A_N at p. The curvature tensor R of M is given by

$$R(X,Y;Z,W) = g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W)),$$

from which the Ricci tensor Ric of M is obtained as

$$Ric(X,Y) = ng(h(X,Y),H) - \Sigma_{i=1}^{n} g(h(X,e_i),h(Y,e_i)), \qquad (3.1)$$

where $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame of M. Sicne M is positively curved and for a unit vector field X, Ric(X,X) is the sum of the sectional curvatures, $Ric(X,X) > 0$. Thus from (3.1) we obtain $g(h(X,X),H) > 0$. This given that $g(A_N X, X) > 0$, for each unit vector field $X \in \mathfrak{X}(M)$. This proves that all the eigenvalues of A_N are positive at each point of M. Thus the height function f_N has no non-degenerate critical points of index $i=1,2,\ldots,n-1$. Using Morse inequalities we obtain

$$H^1(M,R) = 0, \ldots\ldots, H^{n-1}(M,R) = 0.$$

Since M is compact, we get that M is a homology sphere.

COROLLARY 3.1. The real projective space RP^m and the complex projective space CP^2 cannot be isometrically immersed in R^n with parallel mean curvature vector.

Combining Theorem 2.1 with Theorem 3.1, we get

COROLLARY 3.2. There does not exist an n-dimenstional compact and connected positively curved totally real submanifold in R^{2n} with parallel mean curvature vector.

Remark. The Chern-Hopf problem is "The Euler-poincare' characteristic class of an even dimensional positively curved manifold M satisfies $\chi(M) > 0$". For class of even dimensional positively curved compact and connected manifolds which admit isometric commersion in R^n with parallel mean curvature vector we have the following corollary to Theorem 3.1.

COROLLARY 3.3. Let M be an even dimensional compact and connected positively curved submanifold of R^n with parallel mean curvature vector. Then $\chi(M) > 0$.

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