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FACTORIZATION OF k-QUASIHYPONORMAL OPERATORS

S. C. ARORA

Department of Mathematics
University of Delhi
Delhi - 110007
India

and

J. K. THUKRAL

Department of Mathematics
S.R.C.C.
University of Delhi
Delhi - 110007
India

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ABSTRACT. Let A be the class of all operators T on a Hilbert space H such that $R(T^{*^k}T)$, the range space of $T^{*^k}T$, is contained in $R(T^{*^{k+1}})$, for a positive integer k. It has been shown that if $T \in A$, there exists a unique operator C_T on H such that

(i) $T^{*^k}T = T^{*^{k+1}}C_T$;

(ii) $||C_T||^2 = \inf\{\mu: \mu \geq 0 \text{ and } (T^{*^k}T)(T^{*^k}T)^* \leq \mu\, T^{*^{k+1}}\, T^{k+1}\}$;

(iii) $N(C_T) = N(T^{*^k}T)$ and

(iv) $R(C_T) \subseteq \overline{R(T^{k+1})}$

The main objective of this paper is to characterize k-quasihyponormal; normal, and self-adjoint operators T in A in terms of C_T. Throughout the paper, unless stated otherwise, H will denote a complex Hilbert space and T an operator on H, i.e., a bounded linear transformation from H into H itself. For an operator T, we write R(T) and N(T) to denote the range space and the null space of T.

KEY WORDS AND PHRASES. Self-adjoint, normal, unitary, quasinormal, hyponormal, quasihyponormal, k-quasihyponormal, isometry, partial isometry, null space, range space and the projection.
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1. INTRODUCTION

T is said to be quasinormal if $T(T^*T) = (T^*T)T$, hyponormal if $T^*T \geq TT^*$ or equivalently $||T^*x|| \leq ||Tx||$ for each x in H, k-quasihyponormal (Campbell and Gupta

[1]) for a positive integer k if $T^{*k}(T^*T - TT^*)T^k \geq 0$ or equivalently $||T^*T^k x|| \leq ||T^{k+1}x||$ for each x in H.

The purpose of this paper is to consider the class A of those operators T such that $R(T^{*k}T) \subseteq R(T^{*k+1})$ for a positive integer k. More precisely, our aim is to identify those operators T in A which are k-quasihyponormal, normal and self-adjoint. The motivation is due to Embry [2] who considered the class of operators T satisfying $R(T) \subseteq R(T^*)$ and Patel [5] who discussed the class of operators T satisfying $R(T^*T) \subseteq R(T^{*2})$. If $T \in A$, then by Douglas' theorem [3, Theorem 1] there exists a unique operator C_T such that

(i) $T^{*k}T = T^{*k+1}C_T$;

(ii) $||C_T||^2 = \inf\{\mu: \mu > 0 \text{ and } (T^{*k}T)(T^{*k}T)^* \leq \mu T^{*k+1}T^{k+1}\}$;

(iii) $N(C_T) = N(T^{*k}T)$; and

(iv) $R(C_T) \subseteq \overline{R(T^{k+1})}$.

2. MAIN RESULTS.

By Douglas' theorem [3, Theorem 1], the class A contains all k-quasihyponormal operators.

THEOREM 2.1. An operator T in A is k-quasihyponormal if and only if $||C_T|| \leq 1$.

PROOF. If $||C_T|| \leq 1$, $||T^*T^k x|| = ||C_T^* T^{k+1}x|| \leq ||T^{k+1}x||$ for all x in H and hence T is k-quasihyponormal.

Conversely, assume that T is k-quasihyponormal. Since

$$||C_T^* T^{k+1}x|| = ||T^*T^k x|| \leq ||T^{k+1}x||$$

for all x in H, $||C_T^* y|| \leq ||y||$ for all y in $\overline{R(T^{k+1})}$. Also since $R(C_T) \subseteq \overline{R(T^{k+1})}$, i.e. $\overline{R(T^{k+1})}^{\perp} \subseteq N(C_T^*)$, we have $C^*_T x = 0$ for all x in $\overline{R(T^{k+1})}^{\perp}$. Thus for each x in H, $||C_T^* x|| \leq ||x||$ and consequently $||C_T|| = ||C_T^*|| \leq 1$.

To prove our next result, we need the following lemma.

LEMMA 2.1. Let T be a quasinormal operator. Then for any positive integer k

(a) $T^*T^k = T^{k-1}T^*T$

(b) $||(T^*T)^{k/2}x|| = ||T^k x||$ for all vectors x in H

(c) $N(T^{*k}T) \subseteq N(T^{*k})$.

PROOF. (a) We prove it by induction on k. For k = 1, trivial. For k = 2, again it holds since T is quasinormal. Now assume that the result is true for any positive integer m > 2. Then $T^*T^{m+1} = (T^*T^m)T = (T^{m-1}T^*T)T = T^{m-1}(T^*T)T = T^{m-1}TT^*T = T^m T^*T$. Hence by induction the result follows. (b) It is an immediate consequence of the fact that if T is quasinormal, then $(T^*T)^k = T^{*k}T^k$ for any positive integer k. (c)

Let $x \in N(T^{*k}T)$. Then $T^{*k}Tx = 0$, i.e., $T^*T\,T^{*k-1}x = 0$ by (a). Thus $T^{*k-1}x \in N(T^*T) = N(T)$. But $N(T) \subseteq N(T^*)$ since T is quasinormal. Therefore $T^{*k}x = 0$, i.e., $x \in N(T^{*k})$.

By using the lemma we obtain the following

THEOREM 2.2. Let $T \in A$ be a quasinormal operator. Then C_T is a quasinormal partial isometry with $R(C_T) = \overline{R(T^{k+1})}$.

PROOF. We have $||C_T^* T^{k+1}x|| = ||T^* T^k x|| = ||T^{k-1}T^*Tx|| = ||(T^*T)^{k-1/2}\,T^*Tx|| = ||(T^*T)^{k+1/2}x|| = ||T^{k+1}x||$ for any x in H. Thus C_T^* is an isometry on $R(T^{k+1})$. But $\overline{R(T^{k+1})} \supseteq R(C_T) = N(C_T^*)^{\perp}$. Therefore C_T^* hence C_T is a partial isometry. Further, since the initial space of a partial isometry S equals the set of all those vectors x satisfying $||Sx|| = ||x||$ [4, p. 63] and since C_T^* is an isometry on $\overline{R(T^{k+1})}$, therefore $\overline{R(T^{k+1})} \subseteq N(C_T^*)^{\perp}$, the initial space of C_T^*. Hence $\overline{R(T^{k+1})} = N(C_T^*)^{\perp} = \overline{R(C_T)} = R(C_T)$ as $R(C_T)$ is closed.

We now prove that C_T is quasinormal. By making use of Lemma 2.1 again, we see that $N(C_T) = N(T^{*k}T) \subseteq N(T^{*k}) \subseteq N(T^{*k+1}) = N(C_T^*)$ since $R(C_T) = \overline{R(T^{k+1})}$. From this it follows that $N(C_T)^{\perp}$ reduces C_T and since C_T is a partial isometry, C_T is of the form $A \oplus 0$, where A is an isometry. This gives that C_T commutes with $C^*_T C_T$ and hence C_T is quasinormal.

LEMMA 2.2. Let $T \in A$ be such that $R(C_T) = \overline{R(T)}$. Then $N(T^{*k}T) = N(T)$.

PROOF. Since $R(C_T) \subseteq \overline{R(T^{k+1})} \subseteq \dots \subseteq \overline{R(T)}$ and, by hypothesis, $R(C_T) = \overline{R(T)}$, we have $R(C_T) = \overline{R(T^{k+1})} = \dots = \overline{R(T)}$. Thus $N(T^*) = N(T^{*2}) = \dots = N(T^{*k}) = N(T^{*k+1})$. Now, if $x \in N(T^{*k}T)$, then $T^{*k}Tx = 0$, i.e. $Tx \in N(T^{*k}) = N(T^*)$. That means $T^*Tx = 0$ or $x \in N(T^*T) = N(T)$. This completes the proof.

Our next result gives a characterization of normal operators in A.

THEOREM 2.3. An operator T in A is normal if and only if C_T is a normal partial isometry with $R(C_T) = \overline{R(T)}$.

PROOF. Let T be normal. Then by Theorem 2.2, C_T is a partial isometry with $R(C_T) = \overline{R(T^{k+1})}$ and hence $R(C_T) = N(T^{*k+1})^{\perp} = N(T^*)^{\perp} = \overline{R(T)}$. Thus by Lemma 2.2, $N(C_T) = N(T^{*k}T) = N(T)$. Therefore $R(C_T) = \overline{R(T)} = N(T^*)^{\perp} = N(T)^{\perp} = N(C_T)^{\perp} = R(C^*_T)$. Since $C_T^* C_T$ is the projection on $R(C_T^*)$ and $C_T C_T^*$ is the projection on $R(C_T)$, we conclude that $C_T C_T^* = C_T^* C_T$.

Assume on the other hand that C_T is a normal partial isometry with $R(C_T) = R(T)$. Since $R(C_T) \subseteq \overline{R(T^{k+1})} \subseteq \overline{R(T^k)} \subseteq \dots \subseteq \overline{R(T)}$, we have $R(C_T) = \overline{R(T^{k+1})} = \overline{R(T^k)} = \dots = \overline{R(T)}$ and consequently $N(T^*) = N(T^{*2}) = \dots = N(C^*_T) = N(C_T) = N(T^{*k}T) =$

$N(T)$ by Lemma 2.2. Thus $||T^*x|| = ||Tx||$ for each x in $\overline{R(T^k)}^{\perp}$. Further since C^*_T is a partial isometry on $R(C_T) = \overline{R(T^{k+1})}$, we have $||T^*T^kx|| = ||C^*_T T^{k+1}x|| = ||T^{k+1}x||$ for each x in H. Thus $||T^*y|| = ||Ty||$ for each y in $\overline{R(T^k)}$. Hence $||T^*x|| = ||Tx||$ for each x in H, i.e., T is normal.

COROLLARY 2.1. Let $T \in A$. Then T is normal and one-to-one if and only if C_T is a unitary operator with $R(C_T) = \overline{R(T)}$.

PROOF. Suppose T is normal and one-to-one. Then by Theorem 2.3, C_T is a normal partial isometry with $R(C_T) = \overline{R(T)}$. Since $N(C_T) = N(T) = \{0\}$, we have $N(C_T)^{\perp} = H$ and thus C_T is an isometry and consequently C_T is a unitary operator.

Conversely, if C_T is a unitary operator with $R(C_T) = \overline{R(T)}$, T is normal by Theorem 2.3. Also by Lemma 2.2, $N(T) = N(T^{*k}T) = N(C_T) = \{0\}$, therefore T is one-to-one.

The next corollary characterizes self-adjoint operators in A.

COROLLARY 2.2. Let $T \in A$. T is self-adjoint if and only if C_T is the projection on $\overline{R(T)}$.

PROOF. Suppose T is self-adjoint. Then by Theorem 2.3, $R(C_T) = \overline{R(T)} = \overline{R(T^{k+1})}$. Since $T^{*k}T = T^{*k+1} C_T$ and T is self-adjoint, we have $T^{k+1} = T^{k+1}C_T$, i.e., $C^*_T T^{k+1} = T^{k+1}$. This means $C^*_T = I$ on $\overline{R(T^{k+1})} = \overline{R(T)}$. Also $C^*_T = 0$ on $\overline{R(T)}^{\perp}$ as $\overline{R(T)}^{\perp} = \overline{R(C_T)}^{\perp} = N(C^*_T)$. Therefore C_T is the projection on $\overline{R(T)}$.

Assume now that C_T is the projection on $\overline{R(T)}$. Then $R(C_T) = \overline{R(T)}$ and hence by Lemma 2.2, $N(C_T) = N(T^{*k}T) = N(T)$. Also, as in the proof of Theorem 2.3, we have $R(C_T) = \overline{R(T^{k+1})} = \ldots = \overline{R(T)}$ and thus $N(T^*) = N(T^{*2}) = \ldots = N(C^*_T) = N(C_T) = N(T)$. Therefore $T^*x = Tx$ for all x in $\overline{R(T^k)}^{\perp}$. Moreover $T^{*k}T = T^{*k+1}C_T$, implies $T^*T^k = C_T T^{k+1}$ as C_T is self-adjoint. But C_T is the projection on $\overline{R(T)} = \overline{R(T^{k+1})}$, therefore $C_T T^{k+1} = T^{k+1}$, That means $T^*y = Ty$ for all y in $\overline{R(T^k)}$. Thus $T^*x = Tx$ for all x in H or T is self-adjoint.

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