

FUNCTIONS STARLIKE WITH RESPECT TO OTHER POINTS

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ABSTRACT. In [7], Sakaguchi introduce the class of functions starlike with respect to symmetric points. We extend this class. For $0 \leq \beta < 1$, let $S_S^*(\beta)$ be the class of normalised analytic functions f defined in the open unit disc D such that $\operatorname{Re} zf'(z)/(f(z)-f(-z)) > \beta$, for some $z \in D$. In this paper, we introduce 2 other similar classes $S_C^*(\beta)$, $S_{SC}^*(\beta)$ as well as give sharp results for the real part of some function for $f \in S_S^*(\beta)$, $S_C^*(\beta)$ and $S_{SC}^*(\beta)$. The behaviour of certain integral operators are also considered.

KEY WORDS AND PHRASES. *Starlike functions of order β , functions starlike with respect to symmetrical points, close-to-convex, integral operators.*

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1. INTRODUCTION.

Let S be the class of analytic functions f , univalent in the unit disc $D = \{z : |z| < 1\}$, with

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

For $0 \leq \beta < 1$, denote by $S^*(\beta)$, the class of starlike functions of order β . Then $f \in S^*(\beta)$ if, and only if, for $z \in D$,

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta.$$

In [7], Sakaguchi introduced the class S_S^* of analytic functions f , normalised by (1.1) which are starlike with respect to symmetrical points. We begin by defining the class S_S^* , which is contained in K , the class of close-to-convex functions.

DEFINITION 1.

A function $f \in S_S^*$ if, and only if, for $z \in D$,

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)-f(-z)} \right\} > 0.$$

We now extend this definition as follows:

DEFINITION 2.

A function f with normalisations (1.1) is said to be starlike of order β , with respect to symmetric points if, and only if, for $z \in D$ and $0 \leq \beta < 1$,

$$\operatorname{Re} \left(\frac{2zf'(z)}{f(z)-f(-z)} \right) > \beta.$$

We denote this class by $S_S^*(\beta)$ and note that $S_S^* = S_S^*(0)$.

In the same manner, we define the following new classes of close-to-convex functions, which are generalisations of the classes in El-Ashwah and Thomas [2].

DEFINITION 3.

A function f normalised by (1.1) is said to be starlike of order β , with respect to conjugate points if, and only if, for $z \in D$ and $0 \leq \beta < 1$,

$$\operatorname{Re} \left(\frac{2zf'(z)}{f(z)+f(\bar{z})} \right) > \beta.$$

We denote this class by $S_C^*(\beta)$.

DEFINITION 4.

A function f normalised by (1.1) is said to be starlike of order β , with respect to symmetric conjugate points if, and only if, for $z \in D$ and $0 \leq \beta < 1$,

$$\operatorname{Re} \left(\frac{2zf'(z)}{f(z)-f(\bar{z})} \right) > \beta.$$

We denote this class by $S_{sc}^*(\beta)$.

REMARK.

The class S_S^* has been studied by several authors, (eg. Wu [9] and Stankiewicz [8]).

For $f \in S_S^*(\beta)$, Owa et al [4] proved that for $\frac{1}{4} \leq \beta < \frac{1}{2}$,

$$\operatorname{Re} \left(\frac{f(z)-f(-z)}{z} \right) > \frac{2}{3-4\beta}, \quad z \in D.$$

2. RESULTS.

THEOREM 1.

Let $f \in S_S^*(\beta)$, then for $z = re^{i\theta} \in D$,

$$\operatorname{Re} \left(\frac{f(z)-f(-z)}{z} \right)^{1/(1-\beta)} \geq \frac{2^{1/(1-\beta)}}{1+r^2} > 2^{\beta/(1-\beta)}.$$

The result is sharp for f_0 given by

$$f_0(z) - f_0(-z) = 2z(1+z^2)^{\beta-1}.$$

To prove Theorem 1, we first require the following lemma.

LEMMA 1.

Let $g \in S_S^*(\beta)$ and be odd. Then for $z = re^{i\theta} \in D$,

$$\operatorname{Re} \left(\frac{g(z)}{z} \right)^{1/(1-\beta)} \geq \frac{1}{1+r^2}.$$

PROOF OF LEMMA.

Pinchuk [5] showed that if $F \in S^*(\beta)$, then for $z = re^{i\theta} \in D$,

$$\left| \left(\frac{z}{F(z)} \right)^{1/2(1-\beta)} - 1 \right| \leq r. \tag{2.1}$$

Since g is odd, we may write $[g(z)]^2 = F(z^2)$, so that (2.1) gives

$$\left| \left(\frac{z}{g(z)} \right)^{1/(1-\beta)} - 1 \right| \leq r^2,$$

where on squaring both sides, gives

$$\left(\frac{r}{|g(z)|} \right)^{2/(1-\beta)} - 2 \operatorname{Re} \left[\left(\frac{z}{g(z)} \right)^{1/(1-\beta)} \right] + 1 \leq r^4.$$

Thus

$$\begin{aligned} 2 \operatorname{Re} \left[\left(\frac{g(z)}{z} \right)^{1/(1-\beta)} \right] &\geq (1-r^4) \left(\frac{|g(z)|}{r} \right)^{2/(1-\beta)} + 1 \\ &\geq \frac{1-r^4}{(1+r^2)^2} + 1, \end{aligned}$$

where we have used the inequality [6]

$$|g(z)| \geq \frac{r}{(1+r^2)^{1-\beta}},$$

for odd starlike functions of order β .

The Lemma now follows at once.

PROOF OF THEOREM 1.

Since $f \in S_c^*(\beta)$, it follows that we may write

$$g(z) = \frac{f(z) - f(-z)}{2},$$

for g an odd starlike function of order β . An application of Lemma 1 proves the Theorem.

Results analogous to Theorem 1 can also be found for the classes $S_c^*(\beta)$ and $S_{sc}^*(\beta)$.

THEOREM 2.

Let $f \in S_c^*(\beta)$. Then for $z = re^{i\theta} \in D$,

$$\operatorname{Re} \left(\frac{f(z) + \overline{f(\bar{z})}}{z} \right)^{1/2(1-\beta)} \geq \frac{(\sqrt{2})^{1/(1-\beta)}}{1+r} > 2^{(2\beta-1)/2(1-\beta)}.$$

The result is sharp for $f(z) + \overline{f(\bar{z})} = 2z(1+z)^{2(\beta-1)}$

PROOF

Since $f \in S_c^*(\beta)$, it is easy to see that, if

$$F(z) = \frac{f(z) + \overline{f(\bar{z})}}{2}$$

then $F \in S^*(\beta)$. Using the same techniques as in the proof of Lemma 1, it follows from (2.1) that

$$\operatorname{Re} \left(\frac{F(z)}{z} \right)^{1/2(1-\beta)} \geq \frac{1}{1+r}.$$

The result now follows immediately.

Similarly, we have the following result, which we state without proof.

THEOREM 3.

Let $f \in S_{sc}^*(\beta)$. Then for $z = re^{i\theta} \in D$,

$$\operatorname{Re} \left(\frac{f(z) - \overline{f(-\bar{z})}}{z} \right)^{1/2(1-\beta)} \geq \frac{2^{1/2(1-\beta)}}{1+r} > 2^{(2\beta-1)/2(1-\beta)}.$$

The result is sharp for $f(z) - \overline{f(-\bar{z})} = 2z(1+z)^{2(\beta-1)}$

We now consider the results of some integral operators. In [1] Das and Singh, obtained analogous results of the Libera integral operator. They proved that for $f \in S_S^*(0)$, the function h given by

$$h(z) = \frac{1}{2} \int_0^z t^{-1} [f(t) - f(-t)] dt$$

also belongs to $S_S^*(0)$.

The result below generalises that of Das and Singh.

THEOREM 4.

Let $f \in S_S^*(\beta)$. Then the function H defined by

$$H(z) = \frac{a+1}{2z^a} \int_0^z t^{a-1} [f(t) - f(-t)] dt, \quad (2.2)$$

also belongs to $S_S^*(\beta)$ for $z \in D$ and $a + \beta > 0$.

We first require the following Lemma due to Miller and Mocanu [5].

LEMMA 2.

Let M and N be analytic in D with $M(0) = N(0) = 0$ and let β be any real number. If $N(z)$ maps D onto a (possibly many sheeted) region which is starlike with respect to the origin, then for $z \in D$,

$$\operatorname{Re} \frac{M'(z)}{N'(z)} > \beta \implies \operatorname{Re} \frac{M(z)}{N(z)} > \beta,$$

and

$$\operatorname{Re} \frac{M'(z)}{N'(z)} < \beta \implies \operatorname{Re} \frac{M(z)}{N(z)} < \beta.$$

PROOF OF THEOREM 4.

(2.2) gives,

$$\begin{aligned} \frac{2zH'(z)}{H(z) - H(-z)} &= \frac{z^a [f(z) - f(-z)] - a \int_0^z t^{a-1} [f(t) - f(-t)] dt}{\int_0^z t^{a-1} [f(t) - f(-t)] dt} \\ &= \frac{M(z)}{N(z)}, \text{ say.} \end{aligned}$$

Note that $M(0) = N(0) = 0$ and for $f \in S_S^*(\beta)$,

$$\operatorname{Re} \left(1 + \frac{zN''(z)}{N'(z)} \right) = a + \operatorname{Re} \left(\frac{z[f'(z)+f'(-z)]}{f(z)-f(-z)} \right) > a + \beta.$$

Thus $N(z)$ is starlike if, and only if $a > -\beta$.

Furthermore, since

$$\operatorname{Re} \frac{M'(z)}{N'(z)} = \operatorname{Re} \left(\frac{z[f'(z)+f'(-z)]}{f(z)-f(-z)} \right) > \beta.$$

Lemma 2 shows that $H \in S^*(\beta)$.

Finally, we give the following analogous results for the classes $S_c^*(\beta)$ and $S_{sc}^*(\beta)$.

THEOREM 5.

Let $f \in S_c^*(\beta)$. Then H defined by

$$H(z) = \frac{a+1}{2z^a} \int_0^z t^{a-1} [f(t) + \overline{f(\bar{t})}] dt, \tag{2.3}$$

also belongs to $S_c^*(\beta)$ for $z \in D$ and $a + \beta > 0$.

PROOF.

Since $f \in S_c^*(\beta)$, (2.3) gives

$$\begin{aligned} \overline{\int_0^{\bar{z}} t^{a-1} [f(t) + \overline{f(\bar{t})}] dt} &= 2 \int_0^{\bar{z}} t^{a-1} \left(t + \sum_{n=2}^{\infty} \operatorname{Re} a_n t^n \right) dt \\ &= 2 \int_0^z t^{a-1} \left(t + \sum_{n=2}^{\infty} \operatorname{Re} a_n t^n \right) dt \\ &= \int_0^z t^{a-1} [f(t) + \overline{f(\bar{t})}] dt. \end{aligned}$$

Thus

$$\begin{aligned} \frac{2z H'(z)}{H(z) + H(\bar{z})} &= \frac{z^a [f(z) + \overline{f(\bar{z})}] - a \int_0^z t^{a-1} [f(t) + \overline{f(\bar{t})}] dt}{\int_0^z t^{a-1} [f(t) + \overline{f(\bar{t})}] dt} \\ &= \frac{M(z)}{N(z)}, \end{aligned}$$

where $M(0) = N(0) = 0$ and $N \in S^*$ for $a + \beta > 0$.

On using Lemma 2 it follows that $H \in S_c^*(\beta)$.

THEOREM 6.

Let $f \in S_{sc}^*(\beta)$. Then H defined by

$$H(z) = \frac{a+1}{2z^a} \int_0^z t^{a-1} [f(t) - \overline{f(-\bar{t})}] dt, \tag{2.4}$$

also belongs to $S_{sc}^*(\beta)$ for $z \in D$ and $a + \beta > 0$.

PROOF.

For $f \in S_{sc}^*(\beta)$, (2.4) gives

$$\begin{aligned} \overline{H(-\bar{z})} &= \frac{a+1}{(-z)^a} \int_0^{-\bar{z}} t^{a-1} [f(t) - \overline{f(-\bar{t})}] dt \\ &= \frac{a+1}{(-z)^a} \left\{ \frac{2(-z)^{a+1}}{a+1} + \sum_{n=2}^{\infty} \frac{(-z)^{n+a}}{n+a} (\overline{a_n} + (-1)^{n+1} a_n) \right\} \\ &= \frac{-(a+1)}{z^a} \int_0^z t^{a-1} [f(t) - \overline{f(-\bar{t})}] dt. \end{aligned}$$

As before, writing

$$\frac{2zH'(z)}{H(z) - \overline{H(-\bar{z})}} = \frac{M(z)}{N(z)},$$

one can show that $N \in S^*$ and hence using Lemma 2 the result follows.

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