

## ON THE INTEGRABILITY OF A K-CONFORMAL KILLING EQUATION IN A KAEHLERIAN MANIFOLD

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(Received February 16, 1990)

**Abstract.** We show that a necessary and sufficient condition in order that K-conformal Killing equation is completely integrable is that the Kaehlerian manifold  $K^{2m}$  ( $m > 2$ ) is of constant holomorphic sectional curvature.

**Keywords and phrases.** Kaehlerian manifold, K-conformal Killing equation, completely integrability.

1980 Mathematics Subject Classification Codes. Primary 53C55; Secondary 58G30.

**§1. Introduction.** Let  $M^n$  be an  $n$ -dimensional Riemannian manifold. Denote respectively by  $g_{ji}$ ,  $R_{kji}$ ,  $R_{ji} = R_{rji}$  and  $R = g^{ji}R_{ji}$  the metric, the curvature tensor, the Ricci tensor, and the scalar curvature of Riemannian manifold in terms of local coordinates  $\{x^i\}$ , where Latin indices run over the range  $\{1, 2, \dots, n\}$ .

In  $M^n$ , a  $p$ -form  $u$  is said to be Killing, if it satisfies the Killing-Yano's equation:

$$\nabla_{i_0} u_{i_1 i_2 \dots i_p} + \nabla_{i_1} u_{i_0 i_2 \dots i_p} = 0,$$

where  $\nabla$  denotes the operator of the Riemannian covariant derivative.

The following theorem is well known:

**Theorem A** ([1], [5]). *A necessary and sufficient condition in order that the Killing-Yano's equation is completely integrable is that the Riemannian manifold  $M^n$  ( $n > 2$ ) is space of constant curvature.*

A Riemannian manifold  $M^n$  is called a Sasakian manifold if it admits a unit special Killing 1-form  $\eta$  with constant 1 such that

$$\nabla_i \phi_{ji} = \eta_j g_{ki} - \eta_i g_{kj}, \quad \phi_{kj} = \nabla_k \eta_j.$$

In a Sasakian manifold  $M^n$ , a 1-form  $u$  is called D-Killing if it satisfies the D-Killing equation of type  $\alpha$ :

$$\nabla_j u_i + \nabla_i u_j = -2\alpha u_r (\phi_j{}^r \eta_i + \phi_i{}^r \eta_j),$$

where  $\alpha$  is constant.

Then it is known that

**Theorem B** ([8]). *A necessary and sufficient condition in order that the D-Killing equation of type  $\alpha$  is completely integrable is that the Sasakian manifold  $M^n$  ( $n > 3$ ) is a space of constant  $\phi$ -holomorphic sectional curvature with  $H = 1 - 4\alpha$ .*

We consider the analogy of Theorem A and B in a Kaehlerian manifold, namely, the purpose of this paper is to prove the followings.

**Theorem 3.1.** *If there exists (locally) a K-conformal Killing 2-form  $u_{ji}$  satisfying  $u_{ji}(p) = C_{ji}$  for any point  $p$  of a Kaehlerian manifold  $K^{2m}$  ( $m > 2$ ) and any constants  $C_{ji} (= -C_{ij})$ , then  $K^{2m}$  is of constant holomorphic sectional curvature.*

**Theorem 4.1.** *A necessary and sufficient condition in order that K-conformal Killing equation is completely integrable is that the Kaehlerian manifold  $K^{2m}$  ( $m > 2$ ) is of constant holomorphic sectional curvature.*

We shall recall a K-conformal Killing 2-form and an HP-Killing 1-form in §2. In §3 we shall give the proof of Theorem 3.1. Moreover §4 will be devoted to the integrability condition of the K-conformal Killing equation, that is, the proof of Theorem 4.1 will be given.

§2. Pleriminaries. We represent tensors by their components with respect to the natural basis and use the summation convention. For a differential p-form

$$u = \frac{1}{p!} u_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

with skew symmetric coefficients  $u_{i_1 \dots i_p}$ , the coefficients of its exterior differential  $du$  and the exterior codifferential  $\delta u$  are given by

$$(du)_{i_1 \dots i_{p+1}} = \sum_{a=1}^{p+1} (-1)^{a+1} \nabla_{i_a} u_{i_1 \dots \hat{i}_a \dots i_{p+1}},$$

$$(\delta u)_{i_2 \dots i_p} = -\nabla^r u_{r i_2 \dots i_p},$$

where  $\nabla^h = g^{hj} \nabla_j$  and  $\hat{i}_a$  means  $i_a$  to be deleted.

A Kaehlerian manifold  $K^{2m}$  with metric  $g$  of real dimension  $n = 2m$  is  $M^n$  admitting a parallel tensor field  $\phi_i^h$  such that

$$\phi_i^r \phi_r^j = -\delta_i^j, \quad \phi_{ji} = -\phi_{ij},$$

where we put  $\phi_{ji} = \phi_j^r g_{ri}$ . A Kaehlerian manifold is called of constant holomorphic sectional curvature if the curvature tensor satisfies the following equation:

$$R_{hji}^r = \frac{R}{n(n+2)} (g_{ji} \delta_h^r - g_{hi} \delta_j^r + \phi_{ji} \phi_h^r - \phi_{hi} \phi_j^r - 2\phi_{hj} \phi_i^r).$$

In the sequel, we shall consider a Kaehlerian manifold  $K^{2m}$  and assume that  $m > 1$ .

Now, we want to recall the operator for differential forms in  $K^{2m}$ . Denote by  $\mathcal{F}^p$  the set of all p-forms on  $K^{2m}$ . The operators  $\Gamma : \mathcal{F}^p \rightarrow \mathcal{F}^{p+1}$  and  $\Phi : \mathcal{F}^p \rightarrow \mathcal{F}^p$  are defined by

$$(\Gamma u)_{i_0 \dots i_p} = \sum_{a=0}^p (-1)^a \phi_{i_a}^r \nabla_r u_{i_0 \dots \hat{i}_a \dots i_p},$$

$$(\Phi u)_{i_1 \dots i_p} = \sum_{a=1}^p \phi_{i_a}^r u_{i_1 \dots \hat{i}_a \dots i_p}$$

for any p-form  $u$ . For 0-form  $u_0$ , we defined  $\Phi u_0 = 0$ . In  $K^{2m}$ , the curvature tensor and the Ricci tensor satisfy ([2])

$$\phi_h^r R_{rjk} = \phi_i^r R_{r h j k}, \quad R_{hijk} = \phi_h^r \phi_i^s R_{r s j k}, \tag{2.1}$$

$$\frac{1}{2} \phi^{rs} R_{r s j i} = \phi^{rs} R_{r j s i} = -S_{ji} = S_{ij}. \tag{2.2}$$

where we put  $S_{ji} = \phi_j^r R_{ri}$ .

A 2-form  $u$  will be called K-conformal Killing, if it satisfies

$$\nabla_k u_{ji} + \nabla_j u_{ki} = 2\rho_i g_{kj} - \rho_k g_{ji} - \rho_j g_{ki} + 3(\bar{\rho}_k \phi_{ji} + \bar{\rho}_j \phi_{ki}), \tag{2.3}$$

where we put

$$\rho_i = -\frac{1}{n+2} (\delta u)_i, \quad \bar{\rho}_i = (\Phi \rho)_i.$$

It is easy to see from (2.3) that

$$\phi_h^r \nabla_r u_{ji} + \nabla_k (\Phi u)_{ji} = (\Gamma u)_{hji} - 2\rho_h \phi_{ji} - \rho_j \phi_{ki} + \rho_i \phi_{kj} - \bar{\rho}_j g_{hi} + \bar{\rho}_i g_{kj}. \tag{2.4}$$

By interchanging alternatively indices as  $k \rightarrow j \rightarrow i$  at (2.4) and adding all together, we have

$$d\Phi u = 2\Gamma u. \tag{2.5}$$

Let  $v^\#$  be the vector field obtained from 1-form  $v = v_i dx^i$  by virtue of the metric tensor field  $g$ . Then we have  $v^\# = (v^i)$ . We denote  $\mathcal{L}(v^\#)$  by the Lie derivative with respect to the vector field  $v^\#$ . Then we have

$$\mathcal{L}(v^\#) \left\{ \begin{matrix} i \\ k_j \end{matrix} \right\} = \nabla_h \nabla_j v^i + v^r R_{rkj}^i, \tag{2.6}$$

where  $\left\{ \begin{matrix} i \\ k_j \end{matrix} \right\}$  denotes the Christoffel's symbols. Let  $v^\#$  be an HP-Killing vector field. Then we have ([10])

$$\mathcal{L}(v^\#) \left\{ \begin{matrix} i \\ k_j \end{matrix} \right\} = -\frac{1}{n+2} [(d\delta v)_k \delta_j^i + (d\delta v)_j \delta_k^i - (\Phi d\delta v)_k \phi_j^i - (\Phi d\delta v)_j \phi_k^i],$$

which together with (2.6) satisfies that

$$\nabla_h \nabla_j v_i + v_r R_{hji}^r = -\frac{1}{n+2} [(d\delta v)_k g_{ji} + (d\delta v)_j g_{ki} - (\Phi d\delta v)_k \phi_{ji} - (\Phi d\delta v)_j \phi_{ki}].$$

In the following, any 1-form  $v$  satisfying the above equation is said to be an HP-Killing 1-form.

For an HP-Killing 1-form and a K-conformal Killing 2-form, it is known that the following holds:

**Theorem C ([9]).** *In a Kaehlerian manifold  $K^{2m}$ , for an HP-Killing 1-form  $v$ ,  $d\Phi v$  is a closed K-conformal Killing 2-form.*

**§3. Proof of Theorem 3.1.** At first, for a K-conformal Killing 2-form  $u_{ji}$ , the following equations hold ([7]):

$$\begin{aligned} &R_{jkl}^r u_{ir} + R_{ilk}^r u_{jr} + R_{lij}^r u_{kr} + R_{kji}^r u_{lr} \\ &= (\rho_{ij} + \rho_{ji})g_{kl} - (\rho_{ik} + \rho_{ki})g_{jl} + (\rho_{kl} + \rho_{lk})g_{ij} \\ &- (\rho_{jl} + \rho_{lj})g_{ik} + (\tilde{\rho}_{ij} - \tilde{\rho}_{ji})\phi_{kl} - (\tilde{\rho}_{ik} - \tilde{\rho}_{ki})\phi_{jl} \\ &+ (\tilde{\rho}_{kl} - \tilde{\rho}_{lk})\phi_{ij} - (\tilde{\rho}_{jl} - \tilde{\rho}_{lj})\phi_{ik} + 2(\tilde{\rho}_{kj} - \tilde{\rho}_{jk})\phi_{il} \\ &+ 2(\tilde{\rho}_{il} - \tilde{\rho}_{li})\phi_{kj}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \rho_{ij} + \rho_{ji} &= \frac{1}{(n-2)(n+4)} [(n+1)(R_i{}^r u_{jr} + R_j{}^r u_{ir}) \\ &- 3(\phi_i{}^e S_j{}^r + \phi_j{}^e S_i{}^r)u_{er}], \end{aligned} \tag{3.2}$$

$$\begin{aligned} \tilde{\rho}_{ij} - \tilde{\rho}_{ji} &= \frac{1}{(n+2)(n+4)} [(n+3)(S_i{}^r u_{jr} + S_j{}^r u_{ir}) \\ &+ (\phi_i{}^e R_j{}^r - \phi_j{}^e R_i{}^r)u_{er}], \end{aligned} \tag{3.3}$$

where we put

$$\rho_{ji} = \nabla_j \rho_i, \quad \tilde{\rho}_{ji} = \nabla_j \tilde{\rho}_i.$$

Because of (3.2) and (3.3), equation (3.1) is rewritten as follows:

$$R_{jkl}^r u_{ir} + R_{ilk}^r u_{jr} + R_{lij}^r u_{kr} + R_{kji}^r u_{lr} = A_{lkji} + B_{lkji}, \tag{3.4}$$

where we put

$$\begin{aligned} A_{lkji} &= \frac{1}{(n-2)(n+4)} [(n+1)(R_i{}^r u_{jr} + R_j{}^r u_{ir})g_{kl} - 3(\phi_i{}^e S_j{}^r + \phi_j{}^e S_i{}^r)g_{kl}u_{er} \\ &- (n+1)(R_i{}^r u_{kr} + R_k{}^r u_{ir})g_{jl} + 3(\phi_i{}^e S_k{}^r + \phi_k{}^e S_i{}^r)g_{jl}u_{er} \\ &+ (n+1)(R_k{}^r u_{lr} + R_l{}^r u_{kr})g_{ij} - 3(\phi_k{}^e S_l{}^r + \phi_l{}^e S_k{}^r)g_{ij}u_{er} \\ &- (n+1)(R_j{}^r u_{lr} + R_l{}^r u_{jr})g_{ik} + 3(\phi_j{}^e S_l{}^r + \phi_l{}^e S_j{}^r)g_{ik}u_{er}], \end{aligned}$$

$$\begin{aligned}
B_{lkhji} = & \frac{1}{(n+2)(n+4)} [(n+3)(S_i{}^\tau u_{rj} + S_j{}^\tau u_{ir})\phi_{hl} + (\phi_i{}^\epsilon R_j{}^\tau - \phi_j{}^\epsilon R_i{}^\tau)\phi_{hl}u_{er} \\
& - (n+3)(S_i{}^\tau u_{rk} + S_k{}^\tau u_{ir})\phi_{jl} - (\phi_i{}^\epsilon R_k{}^\tau - \phi_k{}^\epsilon R_i{}^\tau)\phi_{jl}u_{er} \\
& + (n+3)(S_k{}^\tau u_{rl} + S_l{}^\tau u_{kr})\phi_{ij} + (\phi_k{}^\epsilon R_l{}^\tau - \phi_l{}^\epsilon R_k{}^\tau)\phi_{ij}u_{er} \\
& - (n+3)(S_j{}^\tau u_{rl} + S_l{}^\tau u_{jr})\phi_{ih} - (\phi_j{}^\epsilon R_l{}^\tau - \phi_l{}^\epsilon R_j{}^\tau)\phi_{ih}u_{er} \\
& + 2(n+3)(S_k{}^\tau u_{rj} + S_j{}^\tau u_{kr})\phi_{il} + 2(\phi_k{}^\epsilon R_j{}^\tau - \phi_j{}^\epsilon R_k{}^\tau)\phi_{il}u_{er} \\
& + 2(n+3)(S_i{}^\tau u_{rl} + S_l{}^\tau u_{ir})\phi_{hj} + 2(\phi_i{}^\epsilon R_l{}^\tau - \phi_l{}^\epsilon R_i{}^\tau)\phi_{hj}u_{er}].
\end{aligned}$$

By our assumption, we have from (3.4)

$$\begin{aligned}
& R_{jkl}{}^\tau \delta_i{}^\epsilon + R_{ilk}{}^\tau \delta_j{}^\epsilon + R_{lij}{}^\tau \delta_k{}^\epsilon + R_{kji}{}^\tau \delta_l{}^\epsilon \\
& - R_{jkl}{}^\epsilon \delta_i{}^\tau - R_{ilk}{}^\epsilon \delta_j{}^\tau - R_{lij}{}^\epsilon \delta_k{}^\tau - R_{kji}{}^\epsilon \delta_l{}^\tau \\
& = P_{lkji}{}^\tau{}^\epsilon + Q_{lkji}{}^\tau{}^\epsilon,
\end{aligned}$$

where we put

$$\begin{aligned}
P_{lkji}{}^\tau{}^\epsilon = & \frac{1}{(n-2)(n+4)} [(n+1)(R_i{}^\tau \delta_j{}^\epsilon + R_j{}^\tau \delta_i{}^\epsilon - R_i{}^\epsilon \delta_j{}^\tau - R_j{}^\epsilon \delta_i{}^\tau)g_{kl} \\
& - 3(\phi_i{}^\epsilon S_j{}^\tau + \phi_j{}^\epsilon S_i{}^\tau - \phi_i{}^\tau S_j{}^\epsilon - \phi_j{}^\tau S_i{}^\epsilon)g_{kl} \\
& - (n+1)(R_i{}^\tau \delta_k{}^\epsilon + R_k{}^\tau \delta_i{}^\epsilon - R_i{}^\epsilon \delta_k{}^\tau - R_k{}^\epsilon \delta_i{}^\tau)g_{jl} \\
& + 3(\phi_i{}^\epsilon S_k{}^\tau + \phi_k{}^\epsilon S_i{}^\tau - \phi_i{}^\tau S_k{}^\epsilon - \phi_k{}^\tau S_i{}^\epsilon)g_{jl} \\
& + (n+1)(R_k{}^\tau \delta_l{}^\epsilon + R_l{}^\tau \delta_k{}^\epsilon - R_k{}^\epsilon \delta_l{}^\tau - R_l{}^\epsilon \delta_k{}^\tau)g_{ij} \\
& - 3(\phi_k{}^\epsilon S_l{}^\tau + \phi_l{}^\epsilon S_k{}^\tau - \phi_k{}^\tau S_l{}^\epsilon - \phi_l{}^\tau S_k{}^\epsilon)g_{ij} \\
& - (n+1)(R_j{}^\tau \delta_l{}^\epsilon + R_l{}^\tau \delta_j{}^\epsilon - R_j{}^\epsilon \delta_l{}^\tau - R_l{}^\epsilon \delta_j{}^\tau)g_{ik} \\
& + 3(\phi_j{}^\epsilon S_l{}^\tau + \phi_l{}^\epsilon S_j{}^\tau - \phi_j{}^\tau S_l{}^\epsilon - \phi_l{}^\tau S_j{}^\epsilon)g_{ik}],
\end{aligned}$$

$$\begin{aligned}
Q_{lkji}{}^\tau{}^\epsilon = & \frac{1}{(n+2)(n+4)} [(n+3)(\delta_i{}^\epsilon S_j{}^\tau - \delta_j{}^\epsilon S_i{}^\tau - \delta_i{}^\tau S_j{}^\epsilon + \delta_j{}^\tau S_i{}^\epsilon)\phi_{kl} \\
& + (\phi_i{}^\epsilon R_j{}^\tau - \phi_j{}^\epsilon R_i{}^\tau - \phi_i{}^\tau R_j{}^\epsilon + \phi_j{}^\tau R_i{}^\epsilon)\phi_{kl} \\
& - (n+3)(\delta_i{}^\epsilon S_k{}^\tau - \delta_k{}^\epsilon S_i{}^\tau - \delta_i{}^\tau S_k{}^\epsilon + \delta_k{}^\tau S_i{}^\epsilon)\phi_{jl} \\
& - (\phi_i{}^\epsilon R_k{}^\tau - \phi_k{}^\epsilon R_i{}^\tau - \phi_i{}^\tau R_k{}^\epsilon + \phi_k{}^\tau R_i{}^\epsilon)\phi_{jl} \\
& + (n+3)(\delta_k{}^\epsilon S_l{}^\tau - \delta_l{}^\epsilon S_k{}^\tau - \delta_k{}^\tau S_l{}^\epsilon + \delta_l{}^\tau S_k{}^\epsilon)\phi_{ij} \\
& + (\phi_k{}^\epsilon R_l{}^\tau - \phi_l{}^\epsilon R_k{}^\tau - \phi_k{}^\tau R_l{}^\epsilon + \phi_l{}^\tau R_k{}^\epsilon)\phi_{ij} \\
& - (n+3)(\delta_j{}^\epsilon S_l{}^\tau - \delta_l{}^\epsilon S_j{}^\tau - \delta_j{}^\tau S_l{}^\epsilon + \delta_l{}^\tau S_j{}^\epsilon)\phi_{ih} \\
& - (\phi_j{}^\epsilon R_l{}^\tau - \phi_l{}^\epsilon R_j{}^\tau - \phi_j{}^\tau R_l{}^\epsilon + \phi_l{}^\tau R_j{}^\epsilon)\phi_{ih} \\
& + 2(n+3)(\delta_k{}^\epsilon S_j{}^\tau - \delta_j{}^\epsilon S_k{}^\tau - \delta_k{}^\tau S_j{}^\epsilon + \delta_j{}^\tau S_k{}^\epsilon)\phi_{il} \\
& + 2(\phi_k{}^\epsilon R_j{}^\tau - \phi_j{}^\epsilon R_k{}^\tau - \phi_k{}^\tau R_j{}^\epsilon + \phi_j{}^\tau R_k{}^\epsilon)\phi_{il} \\
& + 2(n+3)(\delta_i{}^\epsilon S_l{}^\tau - \delta_l{}^\epsilon S_i{}^\tau - \delta_i{}^\tau S_l{}^\epsilon + \delta_l{}^\tau S_i{}^\epsilon)\phi_{kj} \\
& + 2(\phi_i{}^\epsilon R_l{}^\tau - \phi_l{}^\epsilon R_i{}^\tau - \phi_i{}^\tau R_l{}^\epsilon + \phi_l{}^\tau R_i{}^\epsilon)\phi_{kj}].
\end{aligned}$$

Transvecting the above equation with  $\delta_e{}^l$ , by virtue of the Bianchi's identity we have

$$\begin{aligned}
(n-1)R_{kji}{}^\tau{}^\epsilon = & \frac{1}{(n^2-4)(n+4)} [(n^3+2n^2-4n+4) \\
& \times (R_{ij}\delta_k{}^\tau - R_{ik}\delta_j{}^\tau + g_{ij}R_k{}^\tau - g_{ik}R_j{}^\tau) \\
& - (n^3-12n+4)(\phi_{ij}S_k{}^\tau - \phi_{ik}S_j{}^\tau + 2\phi_{kj}S_i{}^\tau) \quad (3.5) \\
& - 4(n+1)(S_{ij}\phi_k{}^\tau - S_{ik}\phi_j{}^\tau + 2S_{kj}\phi_i{}^\tau) \\
& - (n+1)(n+2)R(g_{ij}\delta_k{}^\tau - g_{ik}\delta_j{}^\tau) \\
& - (n-2)R(\phi_{ij}\phi_k{}^\tau - \phi_{ik}\phi_j{}^\tau + 2\phi_{kj}\phi_i{}^\tau)].
\end{aligned}$$

If we transvect (3.5) with  $\phi_h{}^k\phi_l{}^j$  and regard to (2.1) and (2.2), then we get

$$\begin{aligned}
 (n-1)R_{hli}{}^r = & \frac{1}{(n^2-4)(n+4)} [(n^3+2n^2-4n+4) \\
 & \times (S_{li}\phi_h{}^r - S_{hi}\phi_l{}^r + \phi_{li}S_h{}^r - \phi_{hi}S_l{}^r) \\
 & - (n^3-12n+4)(g_{ih}R_l{}^r - g_{il}R_h{}^r + 2\phi_{hl}S_i{}^r) \\
 & - 4(n+1)(R_{ih}\delta_l{}^r - R_{il}\delta_h{}^r + 2S_{hl}\phi_i{}^r) \\
 & - (n+1)(n+2)R(\phi_{li}\phi_h{}^r - \phi_{hi}\phi_l{}^r) \\
 & - (n-2)R(g_{ih}\delta_l{}^r - g_{il}\delta_h{}^r + 2\phi_{hl}\phi_i{}^r)].
 \end{aligned}$$

Furthermore, contracting the above equation with  $\delta_r{}^h$ , if  $m > 2$  we obtain

$$R_{li} = \frac{R}{n}g_{li},$$

that is,  $K^{2m}$  is an Einstein manifold. Substituting this into (3.5), we can easily get

$$R_{kji}{}^r = \frac{R}{n(n+2)}(g_{ji}\delta_k{}^r - g_{ki}\delta_j{}^r + \phi_{ji}\phi_k{}^r - \phi_{ki}\phi_j{}^r - 2\phi_{kj}\phi_i{}^r),$$

which means that  $K^{2m}$  is of constant holomorphic sectional curvature. Consequently, we complete the proof of Theorem 3.1.

From Theorem 3.1 and C we can get

**Cororally.** *If there ezists (locally) a 2-form  $(d\Phi v)_{ji}$  obtained from an HP-Killing 1-form  $v$  satisfying  $(d\Phi v)_{ji}(p) = C_{ji}$  for any point  $p$  of a Kaehlerian manifold  $K^{2m}$  ( $m > 2$ ) and any constants  $C_{ji}(= -C_{ij})$ , then  $K^{2m}$  is of constant holomorphic sectional curvature.*

**§4. Proof of Theorem 4.1.** The purpose of this section is to prove Theorem 4.1 stated in §1. Namely we shall show the converse of Theorem 3.1 is true.

In the first place, it is known that

**Theorem D ([5]).** *If a Kaehlerian manifold  $K^{2m}$  is an Einstein manifold, the associated 1-form  $\rho_i$  of K-conformal Killing 2-form is Killing, namely,*

$$\nabla_j \rho_i + \nabla_i \rho_j = 0.$$

Moreover, by virtue of (3.3) we find the following.

**Lemma 4.1.** *If a Kaehlerian manifold  $K^{2m}$  is an Einstein manifold, then we have for the associated 1-form  $\rho$  of K-conformal Killing 2-form  $u$*

$$d\Phi \rho = \frac{R}{n(n+2)}\Phi u,$$

hence,  $\Phi u$  is closed.

In a Kaehlerian manifold  $K^{2m}$  we consider the K-conformal Killing equation as a system of partial differential equations of unknown function  $u_{ji}$ . This system is equivalent to the following system of partial differential equations with unknown functions  $u_{ji}(= -u_{ij})$  and  $u_{kji}(= -u_{kij})$ :

$$u_{kji} + u_{jki} = 2\rho_j g_{kj} - \rho_k g_{ji} - \rho_j g_{ki} + 3(\tilde{\rho}_k \phi_{ji} + \tilde{\rho}_j \phi_{ki}), \tag{4.1}$$

$$\nabla_k u_{ji} = u_{kji}, \tag{4.2}$$

$$\begin{aligned}
\nabla_l u_{kji} &= \frac{1}{2}(R_{jki}{}^r u_{ir} + R_{kil}{}^r u_{jr} + R_{iji}{}^r u_{kr}) \\
&\quad + \frac{1}{2}(\rho_{ki} - \rho_{ik})g_{jl} + \frac{1}{2}(\rho_{jk} - \rho_{kj})g_{li} \\
&\quad + \frac{1}{2}(\rho_{ij} - \rho_{ji})g_{kl} - \rho_{lj}g_{ki} + \rho_{li}g_{kj} \\
&\quad + \frac{1}{2}(\bar{\rho}_{kj} - \bar{\rho}_{jk})\phi_{li} + \frac{1}{2}(\bar{\rho}_{ji} - \bar{\rho}_{ij})\phi_{lk} \\
&\quad + \frac{1}{2}(\bar{\rho}_{ik} - \bar{\rho}_{ki})\phi_{lj} + (\bar{\rho}_{il} - \bar{\rho}_{li})\phi_{kj} \\
&\quad + (\bar{\rho}_{lj} - \bar{\rho}_{jl})\phi_{ki} + (2\bar{\rho}_{lk} + \bar{\rho}_{kl})\phi_{ji}.
\end{aligned} \tag{4.3}$$

We shall show that the system is completely integrable if  $K^{2m}$  is a space of constant holomorphic sectional curvature.

From our assumption, we can replace (4.3) by the following equation:

$$\begin{aligned}
\nabla_l u_{kji} &= \frac{R}{n(n+2)}[g_{lk}u_{ij} + g_{lj}u_{ki} + g_{li}u_{jk} \\
&\quad + \phi_{lk}(\Phi u)_{ji} + \phi_{lj}(\Phi u)_{ik} + \phi_{li}(\Phi u)_{kj} \\
&\quad - \phi_{ji}(\Phi u)_{lk} - \phi_{ik}(\Phi u)_{lj} - \phi_{kj}(\Phi u)_{li} \\
&\quad - \phi_{jk}\phi_l{}^r u_{ir} - \phi_{ki}\phi_l{}^r u_{jr} - \phi_{ij}\phi_l{}^r u_{kr}] \\
&\quad + g_{lk}\rho_{ij} + g_{lj}\rho_{ki} + g_{li}\rho_{jk} + g_{hi}\rho_{jl} + g_{hj}\rho_{li} + 3\bar{\rho}_{lk}\phi_{ji},
\end{aligned} \tag{4.4}$$

where we have used Lemma 4.1 and Theorem D.

The equation obtained from (4.1) by differentiation:

$$\partial_l u_{kji} + \partial_l u_{jki} = \partial_l [2\rho_i g_{kj} - \rho_k g_{ji} - \rho_j g_{ki} + 3(\bar{\rho}_k \phi_{ji} + \bar{\rho}_j \phi_{ki})]$$

is satisfied identically by (4.1), (4.2) and (4.3).

Next, we discuss the integrability condition (4.2):

$$\nabla_l \nabla_k u_{ji} - \nabla_k \nabla_l u_{ji} = -R_{lki}{}^r u_{jr} - R_{lki}{}^r u_{jr}. \tag{4.5}$$

Taking account that  $K^{2m}$  is a space of constant holomorphic sectional curvature, we have

$$\begin{aligned}
&-R_{lki}{}^r u_{jr} - R_{lki}{}^r u_{jr} \\
&= \frac{R}{n(n+2)}[g_{lj}u_{ki} + g_{li}u_{jk} - g_{kj}u_{li} - g_{ki}u_{jl} + \phi_{lj}\phi_k{}^r u_{ir} \\
&\quad + \phi_{li}\phi_k{}^r u_{jr} - \phi_{kj}\phi_l{}^r u_{ri} - \phi_{ki}\phi_l{}^r u_{jr} + 2\phi_{lk}(\Phi u)_{ji}].
\end{aligned}$$

On the other hand, by virtue of (4.2) and (4.4),  $\nabla_l \nabla_k u_{ji} - \nabla_k \nabla_l u_{ji}$  becomes the right hand side of the above equation. Thus (4.5) holds.

The integrability condition of (4.4) is

$$\nabla_m \nabla_l u_{kji} - \nabla_l \nabla_m u_{kji} = -R_{mli}{}^r u_{rji} - R_{mlj}{}^r u_{kri} - R_{mli}{}^r u_{kjr}. \tag{4.6}$$

Since  $K^{2m}$  is a space of constant holomorphic sectional curvature, we can get

$$\begin{aligned}
&-R_{mli}{}^r u_{rji} - R_{mlj}{}^r u_{kri} - R_{mli}{}^r u_{kjr} \\
&= \frac{R}{n(n+2)}[g_{mk}u_{lji} + g_{mj}u_{kli} + g_{mi}u_{kjl} \\
&\quad - g_{lk}u_{mji} - g_{lj}u_{kmi} - g_{li}u_{kjm} \\
&\quad + \phi_l{}^r(\phi_{mk}u_{rji} + \phi_{mj}u_{kri} + \phi_{mi}u_{kjr}) \\
&\quad - \phi_m{}^r(\phi_{lk}u_{rji} + \phi_{lj}u_{kri} + \phi_{li}u_{kjr}) \\
&\quad - 2\phi_{ml}(2\rho_k \phi_{ji} + \rho_j \phi_{ki} - \rho_i \phi_{kj} + \bar{\rho}_j g_{ki} - \bar{\rho}_i g_{kj})],
\end{aligned} \tag{4.7}$$

where we have used (2.4), (2.5), (4.2) and Lemma 4.1.

On the other hand, operating  $\nabla_m$  to (4.4) and owing to (4.1) and (4.2), we

obtain

$$\begin{aligned} \nabla_m \nabla_l u_{kji} = & \frac{R}{n(n+2)} [ -g_{lk} u_{mji} - g_{lj} u_{kmi} - g_{li} u_{kjm} \\ & + \phi_{lk} \nabla_m (\Phi u)_{ji} + \phi_{lj} \nabla_m (\Phi u)_{ik} + \phi_{li} \nabla_m (\Phi u)_{kj} \\ & - \phi_{ji} \nabla_m (\Phi u)_{lk} - \phi_{ik} \nabla_m (\Phi u)_{lj} - \phi_{kj} \nabla_m (\Phi u)_{li} \\ & + \phi_{kj} \phi_l^\top u_{rmi} + \phi_{ik} \phi_l^\top u_{rmj} + \phi_{ji} \phi_l^\top u_{rmk} \\ & + (2\rho_j g_{mk} - \rho_m g_{kj} - \rho_k g_{mi} + 3\tilde{\rho}_k \phi_{mi}) g_{lj} \\ & - (2\rho_j g_{mk} - \rho_m g_{kj} - \rho_k g_{mj} + 3\tilde{\rho}_k \phi_{mj}) g_{li} \\ & + (3\rho_l \phi_{mi} + 2\rho_i \phi_{ml} + \rho_m \phi_{li} + \tilde{\rho}_l g_{mi}) \phi_{kj} \\ & + (3\rho_l \phi_{mj} + 2\rho_j \phi_{ml} + \rho_m \phi_{lj} + \tilde{\rho}_l g_{mj}) \phi_{ik} \\ & + (3\rho_l \phi_{mk} + 2\rho_k \phi_{ml} + \rho_m \phi_{lk} + \tilde{\rho}_l g_{mk} + 3\tilde{\rho}_m g_{lk}) \phi_{ji} ] \\ & + g_{lk} \nabla_m \rho_{ij} + g_{lj} \nabla_m \rho_{ki} + g_{li} \nabla_m \rho_{jk} + g_{ki} \nabla_m \rho_{jl} \\ & + g_{kj} \nabla_m \rho_{li} + 3\phi_{ji} \nabla_m \tilde{\rho}_{lk}. \end{aligned}$$

By interchanging the indices  $m$  and  $l$  in the above equation, subtracting from the original and owing to (2.4), (2.5), (4.1), (4.2), Theorem D, Lemma 4.1 and the Ricci's identity, we find that  $\nabla_m \nabla_l u_{kji} - \nabla_l \nabla_m u_{kji}$  reduces to the right hand side of (4.7). Therefore (4.6) holds. Consequently, we complete the proof of Theorem 4.1.

**Acknowledgment.** The author would like to express his hearty thanks to Professor S. Yamaguchi for his helpful advice. He also would like to acknowledge the encouragement of Professor S. Tachibana.

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