

SUBMANIFOLDS OF EUCLIDEAN SPACE WITH PARALLEL MEAN CURVATURE VECTOR

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ABSTRACT. The object of the paper is to study some compact submanifolds in the Euclidean space R^n whose mean curvature vector is parallel in the normal bundle. First we prove that there does not exist an n -dimensional compact simply connected totally real submanifold in R^{2n} whose mean curvature vector is parallel. Then we show that the n -dimensional compact totally real submanifolds of constant curvature and parallel mean curvature in R^{2n} are flat. Finally we show that compact positively curved submanifolds in R^n with parallel mean curvature vector are homology spheres. The last result in particular for even dimensional submanifolds implies that their Euler-poincare' characteristic class is positive, which for the class of compact positively curved submanifolds admitting isometric immersion with parallel mean curvature vector in R^n , answers the problem of Chern and Hopf.

KEY WORDS AND PHRASES. Submanifolds, totally real submanifolds, homology sphere, Euler-poincare' characteristic.

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1. Let g be the flat metric of R^n , $\bar{\nabla}$ be the corresponding Riemannian connection. If M is a submanifold of R^n with normal bundle ν , then the connection $\bar{\nabla}$ induces the Riemannian connection

∇ on M and the connection ∇^\perp in the normal bundle ν , and we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad X, Y \in \chi(M), \quad N \in \nu \quad (1.1)$$

where the second fundamental form $h(X, Y)$ is related to $A_N X$, by $g(h(X, Y), N) = g(A_N X, Y)$ and $\chi(M)$ is the Lie-algebra of vector fields on M . The mean curvature vector H of M is given by $H = 1/n \sum_{i=1}^n h(e_i, e_i)$, where (e_1, e_2, \dots, e_n) is a local orthonormal frame of M . The mean curvature vector H is said to be parallel if $\nabla_X H = 0$, $X \in \chi(M)$. If $H = 0$ at each point of M , then M is said to be a minimal submanifold. It is known that if M is a compact submanifold of R^{2n} , then M is not a minimal submanifold (cf. [1]).

The even dimensional Euclidean space R^{2n} has complex structure J which is parallel with respect to the connection $\bar{\nabla}$ that is, R^{2n} is a kaehler manifold. A submanifold M of R^{2n} is said to be totally real if $JTM \in \nu$, where TM is the tangent bundle of M . In the case $\dim M = n$ and M is totally real submanifold of R^{2n} , using (1.1), we obtain

$$\nabla_X^\perp JY = J\nabla_X Y \text{ and } h(X, Y) = JA_{JY}X, \quad X, Y \in \chi(M). \quad (1.2)$$

2. In this section we study the n -dimensional totally real submanifold M of R^{2n} with parallel mean curvature vector H , first under the topological restriction on M that, it is compact and its first Betti number is zero, and then under the geometric restriction that it is a space of constant curvature.

THEOREM 2.1. There does not exist an n -dimensional compact totally real submanifold with first Betti number equal to zero and with parallel mean curvature vector in R^{2n} .

PROOF. Let M be an n -dimensional compact totally real submanifold of R^{2n} with parallel mean curvature vector H . Then JH is a parallel vector field on M .

The 1-form η dual to JH is also parallel and hence harmonic. If $H^1(M; \mathbb{R}) = 0$, then η and hence H must vanish. But this would mean that M is a compact minimal submanifold of R^{2n} , which is impossible (cf. [1]).

THEOREM 2.2. Let M be an n -dimensional ($n \geq 2$) compact totally real submanifold of constant curvature in R^{2n} with parallel mean curvature vector. Then M is flat.

PROOF. If the curvature is nonzero constant, then M is irreducible and cannot admit a nonzero parallel vector field JH .

3. In this section we shall be concerned with the positively curved submanifolds with parallel mean curvature vector in R^n . We prove the following.

THEOREM 3.1. Let M be a compact and connected positively curved submanifold with parallel mean curvature vector in R^n . Then M is a homology sphere.

PROOF. Since M is compact, connected and $\int_M H = 0$, the function $\alpha = \|H\|$ is a non-zero constant. Define the unit normal vector field N on M by $N = 1/\alpha H$. If $\phi: M \rightarrow R^n$ is the immersion of M as submanifold of R^n , then the height function $f_N: M \rightarrow R$ is defined by $f_N(p) = g(\phi(p), N)$. The hessian of the height function at a critical point $p \in M$ of f_N is given by the weingarten map A_N at p . The curvature tensor R of M is given by

$$R(X, Y; Z, W) = g(h(Y, Z), h(X, W)) - g(h(X, Z), h(Y, W)),$$

from which the Ricci tensor Ric of M is obtained as

$$\text{Ric}(X, Y) = ng(h(X, Y), H) - \sum_{i=1}^n g(h(X, e_i), h(Y, e_i)), \quad (3.1)$$

where $\{e_1, e_2, \dots, e_n\}$ is a local orthonormal frame of M . Since M is positively curved and for a unit vector field X , $\text{Ric}(X, X)$ is the sum of the sectional curvatures, $\text{Ric}(X, X) > 0$. Thus from (3.1) we obtain $g(h(X, X), H) > 0$. This gives that $g(A_N X, X) > 0$, for each unit vector field $X \in \chi(M)$. This proves that all the eigenvalues of A_N are positive at each point of M . Thus the height function f_N has no non-degenerate critical points of index $i=1, 2, \dots, n-1$. Using Morse inequalities we obtain

$$H^1(M, R) = 0, \dots, H^{n-1}(M, R) = 0.$$

Since M is compact, we get that M is a homology sphere.

COROLLARY 3.1. The real projective space RP^m and the complex projective space CP^2 cannot be isometrically immersed in R^n with parallel mean curvature vector.

Combining Theorem 2.1 with Theorem 3.1, we get

COROLLARY 3.2. There does not exist an n -dimensional compact and connected positively curved totally real submanifold in R^{2n} with parallel mean curvature vector.

Remark. The Chern-Hopf problem is "The Euler-poincare' characteristic class of an even dimensional positively curved manifold M satisfies $\chi(M) > 0$ ". For class of even dimensional positively curved compact and connected manifolds which admit isometric immersion in R^n with parallel mean curvature vector we have the following corollary to Theorem 3.1.

COROLLARY 3.3. Let M be an even dimensional compact and connected positively curved submanifold of R^n with parallel mean curvature vector. Then $\chi(M) > 0$.

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