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PARTITIONING THE POSITIVE INTEGERS WITH HIGHER ORDER RECURRENCES

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ABSTRACT. Associated with any irrational number $\alpha > 1$ and the function $g(n) = [\alpha n + \frac{1}{2}]$ is an array $\{s(i,j)\}$ of positive integers defined inductively as follows: $s(1,1) = 1$, $s(1,j) = g(s(1,j-1))$ for all $j \geq 2$, $s(i,1) =$ the least positive integer not among $s(h,j)$ for $h \leq i-1$ for $i \geq 2$, and $s(i,j) = g(s(i,j-1))$ for $j \geq 2$. This work considers algebraic integers α of degree ≥ 3 for which the rows of the array $s(i,j)$ partition the set of positive integers. Such an array is called a *Stolarsky array*. A typical result is the following (Corollary 2): if α is the positive root of $x^k - x^{k-1} - \ldots - x - 1$ for $k \geq 3$, then $s(i,j)$ is a Stolarsky array.

KEY WORDS AND PHRASES. Stolarsky array, linear recurrence sequence, nearly arithmetic sequence, nearly geometric sequence.
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1. INTRODUCTION.

It is possible to partition the set Z^+ of positive integers as an infinite set of sequences all obeying a common linear recurrence relation. Perhaps the first such array was introduced in 1977 by Stolarsky [1]. The first row of Stolarsky's array is the sequence of Fibonacci numbers, $1, 2, 3, 5, 8, 13, \ldots$ as seen in Table 1. This row and all subsequent rows obey the recurrence $s_j = s_{j-1} + s_{j-2}$ for all $j \geq 3$. Explicitly, each row after the first begins with the least positive integer not in any previous row, and all terms following the first term of a row are then given by the simple *nonlinear* recurrence $s_j = [\alpha s_{j-1} + \frac{1}{2}]$; that is, the integer nearest αs_{j-1}, where α is the positive root of the characteristic polynomial $x^2 - x - 1$ of the *linear* recurrence $s_j = s_{j-1} + s_{j-2}$.

The purpose of this paper is to determine other linear recurrence relations, notably of order ≥ 3, for which the first-order nonlinear recurrence $s_j = [\alpha s_{j-1} + \frac{1}{2}]$, for suitable α, generates an array that partitions Z^+ in the manner of generation of Stolarsky's array. Burke and Bergum [2], Butcher [3], Gbur [4], Hendy [5], Kimberling [6], Morrison [7], and Stolarksy [1] deal with Stolarsky's original array or other arrays with row sequences linearly recurrent with order 2. The results presented here for higher order recurrences are believed to be new.

2. DEFINITIONS AND AN EXAMPLE.

An array $s(i,j)$, $i \geq 1$, $j \geq 1$, of positive integers is a *Stolarsky array* [specifically, a $(c_{k-1}, c_{k-2}, \ldots, c_0)$ *Stolarsky array*] if

(i) every positive integer occurs exactly once in the array, and

(ii) there exist integers $c_{k-1}, c_{k-2}, \ldots, c_0$, where $c_0 \neq 0$ and $k \geq 2$, such that

$$s(i,j) = c_{k-1}s(i,j-1) + c_{k-2}s(i,j-2) + \ldots + c_0 s(i,j-n) \tag{2.1}$$

for all $j \geq k+1$ for all $i \geq 1$.

For a given Stolarsky array, if k is the least positive integer for which (2.1) holds for some choice of integers $c_{k-1}, c_{k-2}, \ldots, c_0$, where $c_0 \neq 0$, then these integers are uniquely determined, and identity (2.1) is the *recurrence of the array*. The array has *order* k, and the polynomial $x_k - c_{k-1}x^{k-1} - c_{k-2}x^{k-2} - \ldots - c_1 x - c_0$ is the *characteristic polynomial of the array*.

1	2	3	5	8	13	21	...
4	6	10	16	26	42	68	...
7	11	18	29	47	76	123	...
9	15	24	39	63	102	165	...
12	19	31	50	81	131	212	...
14	23	37	60	97	157	254	...
17	28	45	73	118	191	309	...

Table 1: The First Seven Rows of Stolarsky's Array

As an example, consider the array in Table 2, found by using the dominant real root $\alpha = 3.62736508471183$ of $x^3 - 3x^2 - 2x - 1$ (in Table 3).

In Table 2, it is easy to verify that the numbers $s(i,j)$ in Row i, for each $i \geq 1$, satisfy the recurrence

$$s(i,j) = 3s(i,j-1) + 2s(i,j-2) + s(i,j-3)$$

for all $j \geq 4$. Now α exceeds 1 and (by Corollary 3) $s(i,j) = [\alpha s(i,j-1) + \frac{1}{2}]$ for all $j \geq 2$ for all $i \geq 1$. Here again note that $s(1,1) = 1$ by definition. It follows that each positive number appears once and only once in the array, as $s(i,1)$ for $i \geq 2$ is, by construction, the least positive integer not among $s(h,j)$ for $1 < h \leq i-1$ and $j \geq 1$. Therefore, the array is a third-order Stolarsky array.

1	4	15	54	196	711	2579	9355	...
2	7	25	91	330	1197	4342	15750	...
3	11	40	145	526	1908	6921	25105	...
5	18	65	236	856	3105	11263	40855	...
6	22	80	290	1052	3816	13842	50210	...
8	29	105	381	1382	5013	18184	65960	...
⋮								

Table 2: A Third-order Stolarsky Array

Our main objective can now be stated as follows: to determine polynomials

$$f(x) = x^k - c_{k-1}x^{k-1} - \ldots - c_1 x - c_0 \tag{2.2}$$

for which the formula $[\alpha n + \frac{1}{2}]$ generates a Stolarsky array in the manner of the above example.

3. CONDITIONS FOR GENERATING STOLARSKY ARRAYS.

LEMMA 1: If $\alpha > 1$ and m and n are positive integers satisfying $m < n$, then $[\alpha m+\frac{1}{2}] < [\alpha n+\frac{1}{2}]$.

PROOF: If $m \le n-1$ then $\alpha m \le \alpha n - \alpha < \alpha n - 1$, so that

$$[\alpha m + \tfrac{1}{2}] \le [\alpha n - 1 + \tfrac{1}{2}] < [\alpha n + \tfrac{1}{2}].$$

LEMMA 2: Suppose the polynomial (2) has a dominant real root $\alpha > 1$. For arbitrary positive integer n, let $g(n) = [\alpha n + \frac{1}{2}], g^2(n) = g(g(n)), \ldots, g^k(n) = g(g^{k-1}(n)), \ldots$. If

$$g^{k+m}(n) = c_{k-1}g^{k+m-1}(n) + \ldots + c_1 g^{m+1}(n) + c_0 g^m(n), \tag{3.1}$$

where $g^0(n) = n$, holds for $m = 0$ and all $n \ge 1$, then (3.1) holds for all $m \ge 0$ for all $n \ge 1$.

PROOF:

$$\begin{aligned} g^{k+m}(n) &= g^k(g^m(n)) \\ &= c_{k-1}g^{k-1}(g^m(n)) + \ldots + c_1 g(g^m(n)) + c_0 g^m(n) \\ &= c_{k-1}g^{k+m-1}(n) + \ldots + c_1 g^{m+1}(n) + c_0 g^m(n). \end{aligned}$$

THEOREM 1: Let $r_1 = ((\alpha n + \frac{1}{2}))$ be the fractional part of $g(n)$ in Lemma 2. That is, $r_1 = \alpha n + \frac{1}{2} - [\alpha n + \frac{1}{2}]$. Let $r_2 = ((\alpha g(n) + \frac{1}{2})), \ldots, r_k = ((\alpha g^{k-1}(n) + \frac{1}{2}))$. Let

$$\begin{aligned} M = {} & \frac{1}{2(\alpha-1)}(c_0 + c_1 + \ldots + c_{k-1} - 1) - \frac{r_1 c_0}{\alpha} - \frac{r_2(c_0 + c_1\alpha)}{\alpha^2} \\ & - \frac{r_3(c_0 + c_1\alpha + c_2\alpha^2)}{\alpha^3} - \ldots - \frac{r_{k-1}(c_0 + c_1\alpha + \ldots + c_{k-2}\alpha^{k-2})}{\alpha^{k-1}} - r_k. \end{aligned}$$

Let $s(1,j) = [\alpha j + \frac{1}{2}]$ for $j \ge 1$ and $s(1,1) = 1$. Define $s(i,j)$ inductively by letting $s(i,1)$ be the least positive integer not among $s(h,j)$ for $1 < h \le i-1$ and $j \ge 1$, and $s(i,j) = [\alpha s(i,j-1) + \frac{1}{2}]$ for $j \ge 2$. Then $\{s(i,j)\}$ is a Stolarsky array if and only if $|M| < 1$.

Before proving Theorem 1, we use the notation introduced there to establish a lemma:

LEMMA 3: $g^i(n) = \alpha^i n + (\frac{1}{2})\frac{\alpha^i - 1}{\alpha - 1} - \sum_{j=1}^{i} r_j \alpha^{i-j}$ for $i \ge 1$.

PROOF: $g(n) = \alpha n + \frac{1}{2} - r_1$, as asserted for $i = 1$. The identity follows by induction on i.

PROOF OF THEOREM 1: In view of Lemmas 1 and 2, it suffices to show that the inequality $|M| < 1$ is equivalent to identity (3.1) for $m = 0$ and all $n \ge 1$.

$$\begin{aligned} & g^k(n) - \sum_{i=1}^{k} c_{k-i} g^{k-i}(n) \\ & = \alpha^k n + \left(\frac{1}{2}\right)\frac{\alpha^k - 1}{\alpha - 1} - \sum_{j=1}^{k} r_j \alpha^{k-j} \\ & - \sum_{i=1}^{k} c_{k-i}\left(\alpha^{k-i} n + (\frac{1}{2})\frac{\alpha^{k-i} - 1}{\alpha - 1} - \sum_{j=1}^{k-i} r_j \alpha^{k-i-j}\right) \end{aligned}$$

$$= nf(\alpha) + \frac{1}{2(\alpha-1)}\left(\alpha^{k-1} - c_{k-1}(\alpha^{k-1}-1) - c_{k-2}(\alpha^{k-2}-1) - \ldots - c_1(\alpha-1)\right)$$
$$-r_1(\alpha^{k-1} - c_{k-1}\alpha^{k-2} - \ldots - c_2\alpha - c_1) - r_2(\alpha^{k-2} - c_{k-1}\alpha^{k-3} - \ldots - c_3\alpha - c_2)$$
$$- \ldots - r_{k-1}(\alpha - c_{k-1}) - r_k,$$

which equals M. Now $g^k(n) - \sum_{i=1}^{k} c_{k-i}g^{k-i}(n)$, as an integral linear combination of integers, is itself an integer. In order for this to be zero, it is necessary and sufficient that $|M| < 1$.

COROLLARY 1: If $c_i \geq 0$, for $0 \leq i \leq k-1$, are integers satisfying

$$c_{k-1} \geq 1 + c_0 + c_1 + \ldots + c_{k-2}, \tag{3.2}$$

then the array $s(i,j)$ is a Stolarsky array.

PROOF: Let $f(x)$ be as in (2.2) with c_i as in (3.2). Since $f(x) > 0$ for all $x \geq c_{k-1}+1$, and $f(c_{k-1}) < 0$, the dominant real root α satisfies $c_{k-1} < \alpha \leq c_{k-1}+1$. Then

$$M < \frac{1}{2(c_{k-1}-1)}(c_0 + c_1 + \ldots + c_{k-2} + c_{k-1} - 1),$$

since each of the numbers

$$\frac{r_1c_0}{\alpha}, \frac{r_2(c_0+c_1\alpha)}{\alpha^2}, \ldots, \frac{r_{k-1}(c_0 + c_1\alpha + \ldots + c_{k-2}\alpha^{k-2})}{\alpha^{k-1}}, r_k$$

is nonnegative and $1 < c_{k-1} < \alpha$.

Consequently, (3.2) implies

$$M < \frac{1}{2(c_{k-1}-1)}(2c_{k-1}-2) = 1.$$

To see that $M > -1$ also, substitute $s_i = 1 - r_i$ for $i = 1, 2, \ldots, k$ to find that

$$M = \frac{-1}{2(\alpha-1)}(c_0 + c_1 + \ldots + c_{k-1} - 1) + \frac{s_1c_0}{\alpha} + \frac{s_2(c_0+c_1\alpha)}{\alpha^2}$$
$$+ \frac{s_3(c_0 + c_1\alpha + c_2\alpha^2)}{\alpha^3} + \ldots + \frac{s_{k-1}(c_0 + c_1\alpha + \ldots + c_{k-2}\alpha^{k-2})}{\alpha^{k-1}} + s_k.$$

Since all these multiples of the s_i are nonnegative,

$$M > -\frac{1}{2(\alpha-1)}(c_0 + c_1 + \ldots + c_{k-1} - 1),$$

so that $M > -1$.

Corollary 1 shows that there exist Stolarsky arrays of every order $k \geq 2$. However, it is possible for $|M|$ to be less than 1 even when inequality (3.2) fails. Corollaries 2 and 3 reveal two such cases.

COROLLARY 2: Let α be the dominant real root of the polynomial

$$p(x) = x^k - x^{k-1} - x^{k-2} - \ldots - x - 1, k \geq 2.$$

Then the Stolarsky array $\{s(i,j)\}$ defined in Theorem 1 is a Stolarsky array having characteristic polynomial $(x-1)p(x)$.

PROOF: Write

$$\begin{aligned}(x-1)p(x) &= (x-1)(x^k - a_{k-1}x^{k-1} - a_{k-2}x^{k-2} - \ldots - a_1x - a_0)\\ &= x^{k+1} - c_kx^k - c_{k-1}x^{k-1} - \ldots - c_1x - c_0,\end{aligned}$$

where $c_k = a_{k-1} + 1, c_{k-1} = a_{k-2} - a_{k-1}, \ldots, c_1 = a_0 - a_1$, and $c_0 = -a_0$.

Then $c_0 + c_1 + \ldots + c_{k-1} + c_k - 1 = 0$, so that

$$M = -\frac{r_1 c_0}{\alpha} - \frac{r_2(c_0 + c_1\alpha)}{\alpha^2} - \frac{r_3(c_0 + c_1\alpha + c_2\alpha^2)}{\alpha^3} - \ldots - \frac{r_k(c_0 + c_1\alpha + \ldots + c_{k-1}\alpha^{k-1})}{\alpha^k} - r_{k+1} \tag{3.3}$$

For the case at hand, namely, $c_0 = -1$ and $c_i = 0$ for $1 \le i \le k-1$, we have

$$M = -r_{k+1} + \sum_{i=1}^{k} \frac{r_i}{\alpha^i} < \frac{1}{\alpha^k} \sum_{i=0}^{k-1} \alpha^i = 1.$$

Also, clearly, $M > -1$. By Theorem 1, $\{s(i,j)\}$ is a Stolarsky array.

COROLLARY 3: Suppose that $p(x) = x^3 - a_2x^2 - a_1x - a_0$ has a dominant real root α that satisfies the inequalities

$$a_0 \ge 1, a_1 \ge a_0(1 - \frac{1}{\alpha}), \text{ and } a_2 \ge (\frac{a_0}{\alpha} + a_1)(1 - \frac{1}{\alpha}).$$

Then the array $\{s(i,j)\}$ is a Stolarsky array.

PROOF: As in the proof of Corollary 2, we have

$$M = \frac{r_1 a_0}{\alpha} + \frac{r_2[a_0 + \alpha(a_1 - a_0)]}{\alpha^2} + \frac{r_3[a_0 + \alpha(a_1 - a_0) + \alpha^2(a_2 - a_1)]}{\alpha^3} - r_4 \tag{3.4}$$

A sufficient condition that $|M| < 1$ is that the coefficients of r_1, r_2, and r_3 be nonnegative. These three inequalities are easily seen to be equivalent to the three stated in the corollary.

4. CONCLUDING REMARKS.

Corollary 3 applies to several cases not previously covered. Following is a table showing several relevant choices of a_2, a_1, a_0, the derived coefficients, and the dominant real root α. Here, the characteristic polynomial is $(x-1)p(x)$, so that $c_3 = a_2 + 1$, $c_2 = a_1 - a_2$, $c_1 = a_0 - a_1$, and $c_0 = -a_0$.

a_2	a_1	a_0	c_3	c_2	c_1	c_0	α
1	1	1	2	0	0	-1	1.83928675521416
2	1	1	3	-1	0	-1	2.54681827688408
2	2	1	3	0	-1	-1	2.83117720720334
3	2	1	4	-1	-1	-1	3.62736508471183

Table 3: Examples for Corollary 3

For example, to generate a $(2,0,0,-1)$ Stolarsky array, let $\alpha = 1.83928675521416$, let $s(i,j) = [\alpha j + \frac{1}{2}]$ for $j \ge 1$, and define $s(i,j)$ via iteration as stated in Theorem 1 with $s(1,1) = 1$. See Table 4.

A notable feature of this array is that the linear recurrence for Row i is given by

$$s(i,j) = 2s(i,j-1) - s(i,j-4) \text{ for } j \ge 5 \text{ and } i \ge 1,$$

and *not* by

$$s(i,j) = s(i,j-1) + s(i,j-2) + s(i,j-3), \tag{4.1}$$

as might have been expected since α is a root of $x^3 - x^2 - x - 1$. To see that (4.1) fails, consider the numbers 8, 15, 28, 52, in Row 4. Of course, $x^4 - 2x^3 + 1$ is a multiple of $x^3 - x^2 - x - 1$. One wonders if there is a Stolarsky array in which at least one row satisfies a second-order recurrence, rows without this property satisfy a third-order recurrence, and the two corresponding characteristic polynomials are relatively prime.

Finally, we note that the arrays $s(i, j)$ investigated in this article have "almost geometric" rows, in the sense that $s(i, j + 1)/s(i, j)$ stays close to α. Moreover, they also have "almost arithmetic" columns. Perhaps someone will wish to investigate these properties further.

1	2	4	7	13	24	44	81	149	274	...
3	6	11	20	37	68	125	230	423	778	...
5	9	17	31	57	105	193	355	653	1201	...
8	15	28	52	96	177	326	600	1104	2031	...
⋮										

Table 4: A $(2, 0, 0, -1)$ Stolarsky Array

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