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TWO-SIDED ESSENTIAL NILPOTENCE

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ABSTRACT. An ideal I of a ring A is essentially nilpotent if I contains a nilpotent ideal N of A such that $J \cap N \neq 0$ whenever J is a nonzero ideal of A contained in I. We show that each ring A has a unique largest essentially nilpotent ideal $EN(A)$. We study the properties of $EN(A)$ and, in particular, we investigate how this ideal behaves with respect to related rings.

KEY WORDS AND PHRASES. Essential ideal, nilpotent ideal, free normalizing extension, crossed product, Morita equivalent, fixed ring.
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1. INTRODUCTION.

Throughout this paper all rings are associative and all ideals are two-sided. The notation $I \triangleleft A$ means that I is an ideal of A.

Let A be a ring and suppose $I \triangleleft A$. If $K \triangleleft A$ and $K \subseteq I$ then K is *A-essential* in I if $0 \neq B \triangleleft A$ and $B \subseteq I$ imply that $B \cap K \neq 0$. The ideal I is an *essentially nilpotent ideal of A* if there is a nilpotent ideal N of A such that $N \subseteq I$ and N is A-essential in I. We shall denote the prime radical of A by $N(A)$. Recall that $N(A)$ is the intersection of the prime ideals of A and that if $I \triangleleft A$, then $N(I) = I \cap N(A)$.

Essential nilpotence was first studied by Fisher [2]. In this paper we show that each ring A contains a unique largest essentially nilpotent ideal which we denote by $EN(A)$. We establish various results concerning this ideal and, in particular, we investigate how this ideal behaves with respect to related rings. For example, we show that $EN(R[x]) = EN(R)[x]$ and that if G is a finite group of automorphisms of R and R has no $|G|$-torsion, then $EN(R*G) = EN(R)*G$.

Proposition 1. *Let $I \triangleleft A$. The following are equivalent.*

1. *I is an essentially nilpotent ideal of A;*
2. *A has an ideal Z such that $Z^2 = 0, Z \subseteq I$ and Z is A-essential in I;*
3. *If $0 \neq K \triangleleft A$ and $K \subseteq I$, then K contains a nonzero nilpotent ideal of A;* and
4. *$N(I)$ is A-essential in I.*

PROOF. 1 *implies* 2. This follows as in [2, Lemma 2.1], but we repeat the argument for the convenience of the reader. Let $\{Z_\lambda : \lambda \in \Lambda\}$ be the collection of all ideals J of A such that $J^2 = 0$ and $J \subseteq I$. Let $\Phi = \{\Gamma \subseteq \Lambda : \Sigma\{Z_\lambda : \lambda \in \Gamma\}$ is direct$\}$. Using Zorn's lemma we may choose M maximal in Φ. Let $Z = \Sigma\{Z_\lambda : \lambda \in M\}$. Then $Z \subseteq I$ and $Z^2 = 0$. Let $B \triangleleft A, B \subseteq I$. If $B \neq 0$ then

$B \cap K \neq 0$ where K is a nilpotent ideal of $A, K \subseteq I$. Thus $B \cap K$ and hence B contains a nonzero ideal J of A such that $J^2 = 0$. The maximality of M ensures that $Z \cap J \neq 0$ and so Z is A-essential in I.

2 *implies* 3. This is clear.

3 *implies* 4. If J is a nilpotent ideal of A and $J \subseteq I$, then $J \subseteq N(I)$ so this implication is also clear.

4 *implies* 1. Since every nonzero ideal of A contained in $N(I)$ contains an ideal J of A with $J^2 = 0$, the argument in the proof that 1 implies 2 shows that there is an ideal Z of A, $Z^2 = 0$, $Z \subseteq N(I)$ and Z is A-essential in $N(I)$. Since $N(I) \triangleleft A$ and $N(I)$ is A-essential in I it follows that Z is A-essential in I.

Let I and J be essentially nilpotent ideals of the ring A. If $0 \neq K \triangleleft A$ and $K \subseteq I + J$, then either $0 \neq KI \subseteq I$ or $0 \neq KJ \subseteq J$ or $K^2 = 0$. In any case, K contains a nonzero nilpotent ideal of A and so $I + J$ is essentially nilpotent by 3 of the above Proposition. A similar argument shows that the sum of all the essentially nilpotent ideals of A is essentially nilpotent. This unique largest essentially nilpotent ideal of A will be denoted by $EN(A)$.

Proposition 2.1. *If θ is an automorphism of A, then $\theta(EN(A)) = EN(A)$.*

2. *For any ring $A, A/EN(A)$ is semiprime. In particular, if $A \triangleleft B$, then $EN(A) \triangleleft B$.*
3. *If $I \triangleleft A, EN(I) = I \cap EN(A)$.*
4. *If $0 \neq e = e^2 \in A$, then $EN(eAe) \subseteq eEN(A)e$.*
5. *If A has an identity, $0 \neq e = e^2 \in A$ and $AeA = A$, then $EN(eAe) = eEN(A)e$.*

PROOF. 1. is clear. For the proof of 2. suppose $EN(A) \subseteq J \triangleleft A$ and $J^2 \subseteq EN(A)$. If $0 \neq K \triangleleft A, K \subseteq J$ then $K^2 = 0$ implies $K \subseteq N(A) \subseteq EN(A)$ and $K^2 \neq 0$ implies $K^2 = K^2 \cap EN(A) \neq 0$. Hence $EN(A)$ is A-essential in J and so J is essentially nilpotent. Hence $J = EN(A)$ and the proof of 2. is complete.

For the proof of 3. we begin by showing that $EN(A) \cap I$ is an essentially nilpotent ideal of I. Let $0 \neq J \triangleleft I, J \subseteq EN(A) \cap I$. In view of 3 of Proposition 1 it is enough to show that J contains a nonzero nilpotent ideal of I. If J is itself nilpotent this is certainly the case. If J is not nilpotent, $J^{*3} \neq 0$ where J^* is the ideal of A which is generated by J. Since $(J^*)^3 \subseteq EN(A)$, $(J^*)^3$ contains a nonzero nilpotent ideal of A and since $(J^*)^3 \subseteq J$ by Andrunakievic's Lemma this completes the proof that $EN(A) \cap I \subseteq EN(I)$.

From 2. we know that $EN(I) \triangleleft A$ and it follows immediately from 4 in Proposition 1 that $EN(I)$ is an essentially nilpotent ideal of A.

To establish 4 we show that $EN(eAe)^*$ is an essentially nilpotent ideal of A where $EN(eAe)^*$ denotes the ideal of A which is generated by $EN(eAe)$. Let $0 \neq J \triangleleft A, J \subseteq EN(eAe)^*$. Then $eJe \subseteq eEN(eAe)^*e \subseteq EN(eAe)$. If $eJe \neq 0, eJe \cap N(eAe) \neq 0$ and so $J \cap N(EN(eAe)^*) \neq 0$ because $N(eAe) = eN(A)e \subseteq N(A)$. If $eJe = 0$, then $J^3 = 0$ and so $J \cap N(EN(eAE)^*) \neq 0$. Thus $N(EN(eAe)^*)$ is A-essential in $EN(eAe)^*$ and this establishes 4.

To prove 5 it suffices to show that $eEN(A)e \subseteq EN(eAe)$, and to do this it is enough to show that $eEN(A)e$ is an essentially nilpotent ideal of eAe. Now $N(eEN(A)e) = eN(EN(A))e = eN(A)e$ and we will show that $eN(A)e$ is eAe-essential in $eEN(A)e$. Let $0 \neq W \triangleleft eAe, W \subseteq eEN(A)e$. Let W^* denote the ideal of A which is generated by W. Since $W^* \subseteq EN(A)$, $K = W^* \cap N(A) \neq 0$. Also, $eKe \subseteq W \cap eN(A)e$ so the proof will be complete if we show that $eKe \neq 0$. If $eKe = 0$, then $AKA = AeAKAeA \subseteq AeKeA = 0$. But since A has an identity and $K \neq 0$, $AKA \neq 0$.

If R and S are rings with the same identity and $R \subseteq S$, then S is a *free normalizing extension* of R and S is a free right and left R-module with a basis X such that $xR = Rx$ for all $x \in X$. Note that in this case each $x \in X$ determines an automorphism θ_x of R defined by $x\theta_x(r) = rx$ for all

$r \in R$. A free normalizing extension S of R satisfies the *essential condition* if whenever $U \subseteq V$ are ideals of S with U S-essential in V and $I \triangleleft R$ such that $IV \neq 0$, then $IV \cap U \neq 0$. If S is a free *centralizing extension* of R; that is, θ_x is the identity automorphism for all $x \in X$, then certainly S satisfies the essential condition because in this case $IV \triangleleft S$. Also, if G is a finite group of automorphisms of R and R has no $|G|$-torsion, then the crossed product $R*G$ satisfies the essential condition. This is because a minor modification of the proof of Lemma 1.2 (ii) in Passman [3] shows that if U and V are ideals of $R*G$ with U $R*G$-essential in V, then U is essential as an $R-R*G$ subbimodule of V.

THEOREM 3. *If S is a free normalizing extension of R which satisfies the essential condition and is such that $N(S) = N(R)S$, then $EN(S) = EN(R)S$.*

PROOF. We first show that $EN(R)S \subseteq EN(S)$. Since $EN(R)$ is invariant under automorphisms of R, $EN(R)S$ is an ideal of S. We show that $N(S)$ is S-essential in $EN(R)S$. Let $0 \neq T \triangleleft S, T \subseteq EN(R)S$ and denote the normalizing basis of S over R by $X = \{x_\lambda : \lambda \in \Lambda\}$. Choose $0 \neq v = \Sigma\{a_\lambda x_\lambda : \lambda \in \Lambda\}$ in T where $a_\lambda \in EN(R)$ and so that v has a minimal number of coefficients not in $N(R)$. Suppose $\delta \in \Lambda$ and $a_\delta \notin N(R)$. Since $0 \neq Ra_\delta R \subseteq EN(R)$ there are $x_j, y_j \in R$ such that $0 \neq \sum_{j=1}^{n} x_j a_\delta y_j \in N(R)$. Then

$$w = \sum_{j=1}^{n} x_j v\theta_\delta(y_j) = \sum_{\lambda \neq \delta} a'_\lambda x_\lambda + \sum_{j=1}^{n} x_j a_\delta x_\delta \theta_\delta(y_j) = \sum_{\lambda \neq \delta} a'_\lambda x_\lambda + \sum_{j=1}^{n} x_j a_\delta y_j x_\delta$$

where the a'_λ are elements of R with the property that $a'_\lambda \in N(R)$ if $a_\lambda \in N(R)$. Since $\sum_{j=1}^{n} x_j a_\delta y_j \neq 0$, $w \neq 0$ and since w has fewer coefficients not in $N(R)$ than does v we have reached a contradiction. It follows that $v \in N(R)S = N(S)$ and hence $N(S)$ is S-essential in $EN(R)S$. Hence $EN(R)S \subseteq EN(S)$.

Suppose that $0 \neq v \in EN(S)$, $v \notin EN(R)S$. Let $v = \sum\{a_\lambda x_\lambda : \lambda \in \Lambda\}$ and assume $\delta \in \Lambda$ is such that $a_\delta \notin EN(R)$. Then $N(R)$ is not R-essential in $(a_\delta) + N(R)$ where (a_δ) denotes the ideal of R which is generated by a_δ. Hence there is an ideal I of R, $0 \neq I \subseteq (a_\delta) + N(R)$ and $I \cap N(R) = 0$. It follows that $IEN(S) \cap N(R)S = 0$ because if $\sum\{r_\lambda x_\lambda : \lambda \in \Lambda, r_\lambda \in R\}$ is in $IEN(S)$ then $r_\lambda \in I$ for all λ. Since I is not nilpotent and $IN(R) \subseteq I \cap N(R) = 0$, $Ia_\delta \neq 0$. Hence $Iv \neq 0$ and so $IEN(S) \neq 0$. Since $N(S)$ is S-essential in $EN(S)$ and S satisfies the essential condition, $IEN(S) \cap N(S) \neq 0$. This contradicts our previous conclusion that $IEN(S) \cap N(R)S = 0$ because $N(S) = N(R)S$. Hence $EN(S) \subseteq EN(R)S$.

It is well-known that if S is a finite normalizing extension of R, then $N(S) \supseteq N(R)$ and so it follows from the proof of the theorem that if S is a finite free normalizing extension of R, then $EN(S) \supseteq EN(R)$.

COROLLARY 4. *If $M_n(A)$ denotes the ring of $n \times n$ matrices with entries from A, then $EN(M_n(A)) = M_n(EN(A))$.*

PROOF. First assume that A has an identity. Since $M_n(A)$ is a free centralizing extension of A and $N(M_n(A)) = M_n(N(A))$ it follows from the theorem that $EN(M_n(A)) = EN(A)M_n(A) = M_n(EN(A))$.

If A does not have an identity, let A' be the usual (Dorroh) unital extension of A. Then from 3 of Proposition 2,

$$\begin{aligned} EN(M_n(A)) &= M_n(A) \cap EN(M_n(A')) \\ &= M_n(A) \cap M_n(EN(A')) \\ &= M_n(A \cap EN(A')) \\ &= M_n(EN(A)). \end{aligned}$$

COROLLARY 5. *If G is a finite group of automorphisms of A and A has no $|G|$-torsion, then $EN(A*G) = EN(A)*G$ where $A*G$ is the crossed product.*

PROOF. As in Corollary 4 we may assume that A has an identity, and it follows from [3, Theorem 2.2] that $N(A*G) = N(A)*G$ so the theorem applies.

COROLLARY 6. $EN(A[x]) = EN(A)[x]$.

PROOF. As above we may assume that A has an identity and [1, Lemma 2L] shows that $N(A[x]) = N(A)[x]$. So, since $A[x]$ is a free centralizing extension of A, the theorem applies.

COROLLARY 7. *If R and S are rings with identity which are Morita equivalent, then R is essentially nilpotent if and only if S is essentially nilpotent.*

PROOF. This follows immediately from 5 of Proposition 2 and Corollary 4.

COROLLARY 8. *Let R be a ring with identity and let G be a finite group of automorphisms of R such that $|G|$ is invertible in R. Then $EN(R^G) \subseteq EN(R)$.*

PROOF. Let $e = |G|^{-1}\sum_{g \in G} g$. Then e is idempotent in the skew group ring $R*G$ and $e(R*G)e = R^G e \cong R^G$. From 4 of Proposition 2, $EN(R^G e) \subseteq EN(R*G)$ and $EN(R*G) = EN(R)*G$ by Corollary 5. Since $EN(R^G e) = EN(R^G)e$ it follows that $EN(R^G) \subseteq EN(R)$.

We note that $EN(R) = R$ does not in general imply that $EN(R^G) \neq 0$. For example, let

$$R = \begin{bmatrix} \mathbf{Q} & \mathbf{Q} \\ 0 & \mathbf{Q} \end{bmatrix}$$

where $\mathbf{Q}$ is the rational field. The cyclic group $G = \{e, \alpha\}$ of order 2 acts as automorphisms of R via

$$\alpha\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & -b \\ 0 & c \end{bmatrix}$$

and

$$R^G = \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & \mathbf{Q} \end{bmatrix}.$$

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