Internat. J. Math. & Math. Sci.
VOL. 15 NO. 3 (1992) 417-424

ON STRICTLY CONVEX AND STRICTLY 2-CONVEX 2-NORMED SPACES II

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(Received October 2, 1990)

ABSTRACT. In this paper a new duality mapping is defined, and it is our object to show that there is a similarity among these three types of characterizations of a strictly convex 2-normed space. This enables us to obtain more new results along each of two types of characterizations. We shall also investigate a strictly 2-convex 2-normed space in terms of the above two different types.

KEY WORDS AND PHRASES: Linear 2-normed space, strict convexity, strict 2-convexity, 2-semi-inner product, bounded linear 2-functional, duality mapping.
1991 AMS SUBJECT CLASSIFICATION CODES. 46B05, 46B10, 46C05.

1. INTRODUCTION.

This article is a continuation of the paper by Lin [11] where we investigated characterizations of strictly convex and strictly 2-convex 2-normed spaces which were initiated by Diminnie, Gähler and White [5,6]. The concept of strictly convex 2-normed space is 2-dimensional analogue of that of strictly convex normed linear space, an important space in functional analysis, and a strictly 2-convex 2-normed space is its natural generalization. A strictly convex 2-normed space is strictly 2-convex (Theorem 8 [6] and Theorem 3 [11]). But the converse is not generally true (Example 2 [6]). Note, however, that strict 2-convexity together with a certain condition is equivalent to strict convexity (Theorem 3 [11]). Most elementary 2-normed spaces originated by Gähler [7] are strictly convex. For example, a 2-normed space of dimension 2, and a 2-inner product space [6]. A strictly convex normed linear space may be characterized in terms of norms by Giles [8], semi-inner products by Berkson [1], or duality mappings by Browder [2], Gudder and Strawther [9] and many others. In this paper a new duality mapping is defined, and it is our object to show that there is a similarity among these three types of characterizations of a strictly convex 2-normed space. This enables us to obtain more new results along each of two types of characterizations. We shall also investigate a strictly 2-convex 2-normed space in terms of the above two different types.

Let X denote a real linear space of dimension greater than one, the following standard definition was introduced in [7]. If $\|.,.\|$ is a real function on $X \times X$, then X is called a 2-normed space with a 2-norm $\|.,.\|$ if the following conditions are satisfied:

(i) $\|x,y\| = 0$ if and only if x and y are linearly dependent;

(ii) $\|x,y\| = \|y,x\|$;

(iii) $\|ax,y\| = |a|\,\|x,y\|$ for any real a; and

(iv) $\|x+y,z\| \le \|x,z\| + \|y,z\|$.

Let X be a 2-normed space throughout this paper. If $x,y,z \in X$ are nonzero vectors, we denote by $V(x)$, $V(x,y)$ and $V(x,y,z)$ the linear manifolds of X generated by x, x and y, x, y and z, respectively.

2. STRICTLY CONVEX 2-NORMED SPACES.

Recall from [5] that X is said to be strictly convex if $\frac{1}{2}\|x+y,z\| = \|x,z\| = \|y,z\| = 1$ for $z \notin V(x,y)$ implies $x = y$. In this section we shall give several characterizations of this space in terms of 2-semi-inner products and duality mappings. But first we need the following lemma which is essential to our consequent theorems, and which is a portion of Theorem 1 in [11] plus three new statements (8), (9), and (10).

LEMMA 1. The following ten statements are equivalent:

(1) X is strictly convex;

(2) $\frac{1}{2}\|x+y,z\| = \|x,z\| = \|y,z\|$ for $z \notin V(x,y)$ implies $x = y$;

(3) $\|x+y,z\| = \|x,z\| + \|y,z\|$ for $z \notin V(x,y)$ implies $x = by$ for some $b > 0$;

(4) $\frac{1}{2}\|x+y,z\| = \|x,z\| = \|y,z\| \neq 0$ for $x \neq y$ implies $z = d(x-y)$ for some $d \neq 0$;

(5) $\|x+ay,z\| = 2\|x,z\|$ for $z \notin V(x,y)$ and $a = \|x,z\|/\|y,z\|$ implies $x = ay$;

(6) $\|x+y,z\| = \|x,z\| + \|y,z\|$ for $z \notin V(x,y)$ implies $\|y,z\|x = \|x,z\|y$;

(7) $\frac{1}{2}\|x+y,z\| = \|x,z\| = \|y,z\| \neq 0$ for $x \neq y$ implies $\|x,y\| \neq 0$ and $z = \pm\|x,z\|(x-y)/\|x,y\|$;

(8) $\|w+x,z\| = \|w+y,z\| \neq 0$ for all $w \in X$ implies $x = y$;

(9) $\|x-y,z\| = |\|x,z\| - \|y,z\||$ for $z \notin V(x,y)$ implies $x = sy$ for some $s > 0$;

(10) $\|x-y,z\| = |\|x,z\| - \|y,z\||$ for $z \notin V(x,y)$ implies $\|y,z\|x = (\|x-y,z\| + \|y,z\|)y$.

PROOF. The equivalence of (1) through (7) was proved in (Theorem 1 [11]), and that (10) $\Rightarrow$ (9) is obvious. That (9) $\Rightarrow$ (3) is clear after we verify the implication (6) $\Rightarrow$ (10).

(6) $\Rightarrow$ (10): We may write the relation in (10) as $\|x,z\| = \|x-y,z\| + \|y,z\|$. So $\|y,z\|(x-y) = \|x-y,z\|y$ by (6) and the result follows.

(2) $\Rightarrow$ (8): Let $w = x$ and $w = y$ in (8), then $\frac{1}{2}\|x+y,z\| = \|x,z\| = \|y,z\|$ for $z \notin V(x,y)$ implies $x = y$ by (2).

(8) $\Rightarrow$ (2): Suppose that $\frac{1}{2}\|x+y,z\| = \|x,z\| = \|y,z\|$ for $z \notin V(x,y)$ and $x \neq y$, then $\|w+x,z\| = \|w+y,z\| \neq 0$ for some $w \in X$ (indeed, $w = x$ and $w = y$) and $x \neq y$, i.e., (8) does not hold.

The concept of 2-semi-inner product defined by Siddiqui and Rizvi [14] is 2-dimensional analogue of that of the usual semi-inner product in functional analysis. A 2-semi-inner product is a mapping $[.,.|.]$ on $X \times X \times X$ into real numbers such that

(i) $[x+x',y\,|\,z] = [x,y\,|\,z] + [x',y\,|\,z]$;

(ii) $[ax,y\,|\,z] = a[x,y\,|\,z]$ for any real a;

(iii) $[x,x\,|\,z] \ge 0$; $[x,x\,|\,z] = 0$ if and only if x and z are linearly dependent; and

(iv) $|[x,y\,|\,z]|^2 \le [x,x\,|\,z][y,y\,|\,z]$.

Every 2-normed space can be made into a 2-semi-inner product space, and the norm is given by $\|x,y\| = [x,x \mid y]^{\frac{1}{2}}$ [14].

THEOREM 1. The following nine statements are equivalent:

(1) X is strictly convex (in the sense of Lemma 1);

(2) $[x,y \mid z] = \|x,z\| = \|y,z\| = 1$ for $z \notin V(x,y)$ implies $x = y$;

(3) $[x,y \mid z] = \|x,z\|^2 = \|y,z\|^2$ for $z \notin V(x,y)$ implies $x = y$;

(4) $[w,x \mid z] = [w,y \mid z]$ for $z \notin V(x,y,w)$ and all $w \in X$ implies $x = y$;

(5) $[ax,y \mid z] = \|x,z\|^2$ for $z \notin V(x,y)$ implies $x = ay$ for some $a > 0$, and $a = 1$ if $\|x,z\| = \|y,z\|$;

(6) $[x,y \mid z] = \|x,z\| \, \|y,z\|$ for $z \notin V(x,y)$ implies $x = ay$ for some $a > 0$;

(7) $[x,y \mid z] = \|x,z\|^2 = \|y,z\|^2 \neq 0$ for $x \neq y$ implies $z = d(x-y)$ for some $d \neq 0$;

(6′) $[x,y \mid z] = \|x,z\| \, \|y,z\|$ for $z \notin V(x,y)$ implies $\|y,z\| x = \|x,z\| y$;

(7′) $[x,y \mid z] = \|x,z\|^2 = \|y,z\|^2 \neq 0$ for $x \neq y$ implies $\|x,y\| \neq 0$ and $z = \pm\|x,z\| \, (x-y)/\|x,y\|$.

PROOF. The following implications are routine: (2) $\Leftarrow$ (5) $\Leftarrow$ (6′) $\Rightarrow$ (6) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7′) $\Rightarrow$ (7). So let us prove that (3) $\Rightarrow$ (4) $\Rightarrow$ (1) $\Rightarrow$ (6′), (2) $\Rightarrow$ (1) $\Rightarrow$ (7′) and (7) $\Rightarrow$ (1).

(1) $\Rightarrow$ (6′): Let $[x,y \mid z] = \|x,z\| \, \|y,z\|$ for $z \notin V(x,y)$, then $(\|x,z\| + \|y,z\|)\|y,z\| = [x+y,y \mid z] \leq \|x+y,z\| \, \|y,z\| \leq (\|x,z\| + \|y,z\|)\|y,z\|$, or $\|x+y,z\| = \|x,z\| + \|y,z\|$. Hence $\|y,z\| x = \|x,z\| y$ by (6) in Lemma 1.

(3) $\Rightarrow$ (4): Let $w = x$ in (4), then $\|x,z\|^2 = [x,y \mid z] \leq \|x,z\| \, \|y,z\|$, or $\|x,z\| \leq \|y,z\|$. If $w = y$, then $\|y,z\| \leq \|x,z\|$ similarly. Hence $\|x,z\| = \|y,z\|$ and $x = y$ by (3).

(4) $\Rightarrow$ (1): Suppose that X is not strictly convex, i.e., $\frac{1}{2}\|x+y,z\| = \|x,z\| = \|y,z\|$ for $z \notin V(x,y)$ and $x \neq y$, we have to show that $[w,x \mid z] = [w,y \mid z]$ for $z \notin V(x,y,w)$ and some z's implies $x \neq y$. Since $\|x,z\| = \|y,z\|$ by the proof (3) $\Rightarrow$ (4) we have $[x,y \mid z] = \|x,z\| \, \|y,z\|$. As in the proof (1) $\Rightarrow$ (6′) we conclude that $\frac{1}{2}\|x+y,z\| = \|x,z\| = \|y,z\|$.

(2) $\Rightarrow$ (1): Let $\frac{1}{2}\|x+y,z\| = \|x,z\| = \|y,z\| = 1$ and $x \neq y$, then, with the aid of the proof (1) $\Rightarrow$ (6′), we can show easily that $[x,y \mid z] = \|x,z\| = \|y,z\| = 1$ implies $x \neq y$.

(1) $\Rightarrow$ (7′): Let $x \neq y$ and $[x,y \mid z] = \|x,z\|^2 = \|y,z\|^2$, so $[x,y \mid z] = \|x,z\| \, \|y,z\|$, then $\frac{1}{2}\|x+y,z\| = \|x,z\| = \|y,z\|$ by the proof (1) $\Rightarrow$ (6′). Hence $\|x,z\| \neq 0$ and $z = \pm\|x,z\| \, (x-y)/\|x,y\|$ by (7) in Lemma 1.

(7) $\Rightarrow$ (1): Suppose by contrapositive that (4) in Lemma 1 does not hold, then by the proof (1) $\Rightarrow$ (6′) it is easily seen that (7) does not hold, and the proof of the theorem is complete.

Motivated by the concepts of bounded linear functionals, and duality mappings on normed linear spaces [2, 9], bounded linear 2-functionals on 2-normed spaces were introduced by White [15], and associated duality mappings were defined in [3]. Let M and N be linear manifolds of X, a bounded linear 2-functional is a mapping f on $M \times N$ into real numbers such that

(i) $f(x+x', y+y') = f(x,y) + f(x,y') + f(x',y) + f(x',y')$;

(ii) $f(ax,by) = abf(x,y)$ for any real numbers a and b; and

(iii) $|f(x,y)| \leq k\|x,z\|$ for some $k \geq 0$ and all $(x,y) \in M \times N$.

In this case the norm of f is defined by

$$\|f\| = \inf\{k: |f(x,y)| \le k\|x,y\|, (x,y) \in M \times N\}.$$

It can be shown that $|f(x,y)| \le \|f\| \|x,y\|$ and $f(x,y) = 0$ if $x \in V(y)$ [15]. We need also a result which is similar to the Hahn-Banach theorem of functional analysis: If $x,z \in X$ and $x \notin V(z)$, then there exists a bounded linear 2-functional f on $X \times V(z)$ such that $f(x,z) = \|x,z\|$ and $\|f\| = 1$ [6, 13, 15].

The following duality mappings defined in [3] are 2-dimensional analogues of usual duality mappings on a normed linear space [2, 9]:

$$I(x,z) = \{f \in X_z^*: f(x,z) = \|f\| \|x,z\|\} \text{ and}$$

$$J(x,z) = \{f \in X_z^*: f(x,z) = \|f\| \|x,z\|, \|f\| = \|x,z\|\}$$

with duality mappings $I,J: X \times V(z) \to 2^{X_z^*}$, where X_z^* is the space of all bounded linear 2-functionals on $X \times V(z)$.

Evidently the following assertions are true: (a) $J(x,z) \subseteq I(x,z)$; (b) $I(x,z) = X_z^*$ if and only if $x \in V(z)$; (c) $I(x,z) = I(cx,dz) = cdI(x,z)$ for $c,d > 0$; (d) $0 \ne f \in I(x,z)$ for $x \notin V(z)$ implies $f \in J(cx,z)$ for some $c > 0$; and (e) If $x \notin V(z)$, then there exists an $f \in J(x,z)$ with $f \ne 0$ (by the Hahn-Banach theorem stated in above).

Let us define another type of duality mapping as follows:

DEFINITION. Let $I'(x,z)$ be the same as $I(x,z)$ which has the following additional properties:

(i) $\|x,z\| \ge \|y,z\|$ if and only if $\|f\| \ge \|g\|$ for $z \notin V(x,y)$, $f \in I(x,z)$ and $g \in I(y,z)$; and

(ii) $\|x,z\| \ge \|x,w\|$ if and only if $\|f\| \ge \|h\|$ for $x \notin V(z,w)$, $f \in I(x,z)$ and $h \in I(x,w)$.

It follows easily from (i) that $f \in I'(x,z) \cap I'(y,z)$ for $z \notin V(x,y)$ if and only if $f(x,z) = \|f\| \|x,z\|$, $f(y,z) = \|f\| \|y,z\|$ and $\|x,z\| = \|y,z\|$. A similar result from (ii) is obtainable.

LEMMA 2. If $0 \ne f \in I'(x,z)$, $0 \ne g \in I'(y,z)$ for $x \ne y$ and $z \notin V(x,y)$, then

(1) $(f-g)(x-y,z) \ge 0$;

(2) $(f-g)(x-y,z) = 0$ if and only if $f(y,z) = \|f\| \|y,z\|$, $g(x,z) = \|g\| \|x,z\|$ and $\|x,z\| = \|y,z\|$;

(3) $(f-g)(x-y,z) = 0$ if and only if $f, g \in I'(x,z) \cap I'(y,z)$.

PROOF. (1) and (2) are straightforward computations and can be found in ([10] p. 379). Indeed, $(f-g)(x-y,z) = (\|f\| - \|g\|)(\|x,z\| - \|y,z\|) + [\|f\| \|y,z\| - f(y,z)] + [\|g\| \|x,z\| - g(x,z)] \ge 0$. (3) is consequences of (2) and a previous remark.

In a similar manner we can prove the following analogous result.

LEMMA 3. If $0 \ne f \in I'(x,z)$, $0 \ne g \in I'(x,w)$ for $z \ne w$ and $x \notin V(z,w)$, then

(1) $(f-g)(x,z-w) \ge 0$;

(2) $(f-g)(x,z-w) = 0$ if and only if $f(x,w) = \|f\| \|x,w\|$, $g(x,z) = \|g\| \|x,z\|$ and $\|x,z\| = \|x,w\|$;

(3) $(f-g)(x,z-w) = 0$ if and only if $f, g \in I'(x,z) \cap I'(x,w)$.

Obviously, I' in Lemma 2 and 3 may be replaced by J. Let # denote the inclusion relation $\subseteq$, $\supseteq$ or $=$.

THEOREM 2. If $x, y \ne 0$, then the following thirteen statements are equivalent:

(1) X is strictly convex (in the sense of Lemma 1);

(2) $I(x,z)\cap I(y,z)\neq\varnothing$ for $z\notin V(x,y)$ implies $x=ay$ for some $a>0$;

(3) $I(x,z)\# I(y,z)$ for $z\notin V(x,y)$ implies $x=ay$ for some $a>0$;

(4) $J(x,z)\cap J(y,z)\neq\varnothing$ for $z\notin V(x,y)$ implies $x=y$;

(5) $J(x,z)J(y,z)$ for $z\notin V(x,y)$ implies $x=y$;

(6) $I'(x,z)\cap I'(y,z)\neq\varnothing$ for $z\notin V(x,y)$ implies $x=y$;

(7) $I'(x,z)I'(y,z)$ for $z\notin V(x,y)$ implies $x=y$;

(8) If $0\neq f\in I'(x,z)$ and $0\neq g\in I'(y,z)$ for $x\neq y$ and $z\notin V(x,y)$, then $(f-g)(x-y,z)>0$;

(9) $J(x,z)\cap J(y,z)\neq\varnothing$ for $x\neq y$ implies $z=d(x-y)$ for some $d\neq 0$;

(2′) $I(x,z)\cap I(y,z)\neq\varnothing$ for $z\notin V(x,y)$ implies $\|y,z\|x=\|x,z\|y$;

(3′) $I(x,z)\# I(y,z)$ for $z\notin V(x,y)$ implies $\|y,z\|x=\|x,z\|y$;

(8′) If $0\neq f\in J(x,z)$ and $0\neq g\in J(y,z)$ for $x\neq y$ and $z\notin V(x,y)$, then $(f-g)(x-y,z)>0$;

(9′) $J(x,z)\cap J(y,z)\neq\varnothing$ for $x\neq y$ implies $\|x,y\|\neq 0$ and $z=\pm\|x,z\|(x-y)/\|x,y\|$.

PROOF. The proof of (2′) $\Rightarrow$ (2) $\Rightarrow$ (3), (2′) $\Rightarrow$ (3′) $\Rightarrow$ (3) and (9′) $\Rightarrow$ (9) are trivial. Equivalences of (1), (4), (5), (6) and (7) are clear after we verify the implications (3) $\Rightarrow$ (1) $\Rightarrow$ (2′). (8′) is, of course, a special case of (8).

(1) $\Rightarrow$ (2′): Let $0\neq f\in I(x,z)\cap I(y,z)=I(x,z)\cap I(\|x,z\|y/\|y,z\|,z)$, then $\|f\|\,\|x+(\|x,z\|y/\|y,z\|),z\|\geq f(x+(\|x,z\|y/\|y,z\|),z)=2\|f\|\,\|x,z\|\geq\|f\|\,\|x+(\|x,z\|y/\|y,z\|),z\|$, or $\|x+(\|x,z\|y/\|y,z\|),z\|=2\|x,z\|$ and hence $\|y,z\|x=\|x,z\|y$ by (5) in Lemma 1.

(3) $\Rightarrow$ (1): Without loss of generality we may assume that $0\neq f\in I(x,z)\subseteq I(y,z)$ in (3). Suppose that $\|x+y,z\|=\|x,z\|+\|y,z\|$ and $x\neq by$ for all $b>0$, i.e., the negation of (3) in Lemma 1, we have to show that $f\in I(x,z)\subseteq I(y,z)$ implies $x\neq by$ for all $b>0$. This follows from the relation $\|f\|\,\|x+y,z\|\geq f(x+y,z)=\|f\|(\|x,z\|+\|y,z\|)\geq\|f\|\,\|x+y,z\|$, or $\|x+y,z\|=\|x,z\|+\|y,z\|$.

(6) $\Rightarrow$ (8): Let $0\neq f\in I'(x,z)$, $0\neq g\in I'(y,z)$, $x\neq y$, $z\notin V(x,y)$ and $(f-g)(x-y,z)=0$, then $f\in I'(x,z)\cap I'(y,z)$ by Lemma 2, and $x\neq y$. Thus (6) does not hold.

(8) $\Rightarrow$ (6): If $f\in I'(x,z)\cap I'(y,z)$ and if $x\neq y$, then $0=(f-f)(x-y,z)>0$ by (8) yielding a contradiction.

(1) $\Rightarrow$ (9′): For $x\neq y$ let $0\neq f\in J(x,z)\cap J(y,z)$, then $\|x,z\|=\|y,z\|=\|f\|\neq 0$. It follows easily that $\frac{1}{2}\|x+y,z\|=\|x,z\|=\|y,z\|\neq 0$. Hence $\|x,y\|\neq 0$ and $z=\pm\|x,z\|(x-y)/\|x,y\|$ by (7) in Lemma 1.

(9) $\Rightarrow$ (1): Consider the negation of (4) in Lemma 1, i.e., $\frac{1}{2}\|x+y,z\|=\|x,z\|=\|y,z\|\neq 0$, $x\neq y$ and $z\neq d(x-y)$ for all $d\neq 0$, then as in the proof (1) $\Rightarrow$ (9′) we can easily conclude that (9) does not hold.

REMARKS. (a) That $J(x,z)\cap J(y,z)\neq\varnothing$ in (9) and (9′) above may be replaced, of course, by $J(x,z)\# J(y,z)$ without any other change in the statements; (b) J in (9) and (9′) may be replaced by I' if $\|x,z\|$ or $\|y,z\|\neq 0$ in addition to the conditions; (c) Though (2) appeared in ([3] Theorem 1), our proof is direct and much simpler. (4) is in ([3] Corollary 3). (8) was discussed in ([10] Theorem 2.5) with a different type of duality mapping; (d) Note that a duality mapping which satisfies the statement (8) is said to be strictly monotone [10] (cf. [2, 9]). In other words, X is strictly convex if and only if I' or J is strictly monotone.

3. STRICTLY 2-CONVEX 2-NORMED SPACES.

According to [6] X is said to be strictly 2-convex if $\|x+z,y+z\|/3=\|x,y\|=\|y,z\|=\|z,x\|=1$ implies $z=x+y$. We now turn to the investigation of this space in terms of 2-semi-inner products and duality mappings. To this end we require first the next result which is a portion of Theorem 2 in [11].

LEMMA 4. The following four statements are equivalent:

(1) X is strictly 2-convex;

(2) $\|x+z,y+z\|=\|x,y\|+\|y,z\|+\|z,x\|$ for $\|x,y\|\,\|y,z\|\,\|z,x\|\neq 0$ implies $z=bx+cy$ for some $b,c>0$;

(3) $\|bx+z,cy+z\|=3\|bx,z\|$ for $\|x,y\|\,\|y,z\|\,\|z,x\|\neq 0$ implies $z=bx+cy$, where $b=\|y,z\|/\|x,y\|$ and $c=\|x,z\|/\|x,y\|$.

(4) $\|x+z,y+z\|=\|x,y\|+\|y,z\|+\|z,x\|$ for $\|x,y\|\,\|y,z\|\,\|z,x\|\neq 0$ implies $z=bx+cy$, where b and c are as in (3).

In order to be able to prove the next theorem we shall use one of the basic properties of a 2-norm that $\|ax+by,y\|=|a|\,\|x,y\|$ for any real numbers a and b [7].

THEOREM 3. The following five statements are equivalent:

(1) X is strictly 2-convex (in the sense of Lemma 4);

(2) $[-x,y\mid y+z]=(\|x,y\|+\|x,z\|)\|y,z\|$ for $\|x,y\|\,\|y,z\|\,\|z,x\|\neq 0$ implies $z=bx+cy$ for some $b,c>0$;

(3) $\frac{1}{2}[-x,y\mid y+z]=\|x,y\|^2=\|y,z\|^2=\|z,x\|^2\neq 0$ implies $z=x+y$;

(4) $\frac{1}{2}[-x,y\mid y+z]=\|x,y\|=\|y,z\|=\|z,x\|=1$ implies $z=x+y$;

(2′) $[-x,y\mid y+z]=(\|x,y\|+\|z,x\|)\|y,z\|$ for $\|x,y\|\,\|y,z\|\,\|z,x\|\neq 0$ implies $z=bx+cy$, where $b=\|y,z\|/\|x,y\|$ and $c=\|x,z\|/\|x,y\|$.

PROOF. The following implications are trivial: (2′) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

(1) $\Rightarrow$ (2′): If $[-x,y\mid y+z]=(\|x,y\|+\|x,z\|)\|y,z\|$, then $(\|x,y\|+\|y,z\|+\|z,x\|)\|y,z\|=[y-x,y\mid y+z]\leq\|y-x,y+z\|\,\|y,z\|=\|(y+z)-(x+z),y+z\|\,\|y,z\|=\|x+z,y+z\|\,\|y,z\|\leq(\|x,y\|+\|y,z\|+\|z,x\|)\|y,z\|$, or $\|x+z,y+z\|=\|x,y\|+\|x,z\|+\|y,z\|$ and the result follows by (4) in Lemma 4.

(4) $\Rightarrow$ (1): If $\|x+z,y+z\|/3=\|x,y\|=\|y,z\|=\|z,x\|=1$ and $z\neq x+y$, we have to show that $\frac{1}{2}[-x,y\mid y+z]=\|x,y\|=\|y,z\|=\|z,x\|=1$ implies $z\neq x+y$. But this is clear from the proof in above.

THEOREM 4. In the following let $I(u,v)$, $J(u,v)$ and $I'(u,v)$ be defined as in the previous section, and let $u\notin V(v)$, then the following seven statements are equivalent:

(1) X is strictly 2-convex (in the sense of Lemma 4);

(2) $I(x,y)\cap I(x,z)\cap I(z,y)\neq\varnothing$ implies $z=bx+cy$ for some $b,c>0$;

(3) $J(x,y)\cap J(x,z)\cap J(z,y)\neq\varnothing$ implies $z=x+y$;

(4) $I'(x,y)\cap I'(x,z)\cap I'(z,y)\neq\varnothing$ implies $z=x+y$;

(5) If $0\neq f\in I'(x,y)$, $0\neq g\in I'(x,z)$ and $0\neq h\in I'(z,y)$ for $z\neq x+y$, then $(f-h)(x-z,y)$ and $(f-g)(x,y-z)>0$;

(2′) $I(x,y)\cap I(x,z)\cap I(z,y)\neq\varnothing$ implies $z=bx+cy$ for $b=\|y,z\|/\|x,y\|$ and $c=\|x,z\|/\|x,y\|$;

(5′) If $0 \neq f \in J(x,y)$, $0 \neq g \in J(x,z)$ and $0 \neq h \in J(z,y)$ for $z \neq x+y$, then $(f-h)(x-z,y)$ and $(f-g)(x,y-z) > 0$.

PROOF. That (2′) ⇒ (2) is trivial. (5′) is a special case of (5), and it is clear that we need to verify that (2) ⇒ (1) ⇒ (2′) and (4) ⇔ (5) only.

(1) ⇒ (2′): Let $0 \neq f \in I(x,y) \cap I(x,z) \cap I(z,y) = I(bx,cy) \cap I(bx,z) \cap I(z,cy)$, where $b = \|y,z\| / \|x,y\|$ and $c = \|x,z\| / \|x,y\|$, then $\|f\| \, \|bx+z,cy+z\| \leq \|f\| \, (\|bx,cy\| + \|bx,z\| + \|z,cy\|) = f(bx+z, cy+z), \leq \|f\| \, \|bx+z,cy+z\|$, or $\|bx+z,cy+z\| = \|bx,cy\| + \|bx,z\| + \|z,cy\| = 3\|bx,z\|$, and $\|x,y\| \, \|y,z\| \, \|z,x\| \neq 0$ by assumption. So $z = bx+cy$ by (3) in Lemma 4.

(2) ⇒ (1): Consider the negation of (2) in Lemma 4, i.e., $\|x+z,y+z\| = \|x,y\| + \|x,z\| + \|z,y\|$, $\|x,y\| \, \|y,z\| \, \|z,x\| \neq 0$ and $z \neq bx+cy$ for all $b,c > 0$, we have to show that $0 \neq f \in I(x,y) \cap I(x,z) \cap I(z,y)$ implies $z \neq bx+cy$ for all $b,c > 0$. This follows from the relation $\|f\| \, \|x+z,y+z\| \geq f(x+z,y+z) = \|f\| \, (\|x,y\| + \|x,z\| + \|z,y\|) \geq \|f\| \, \|x+z,y+z\|$, or $\|x+z,y+z\| = \|x,y\| + \|x,z\| + \|z,y\|$.

(4) ⇒ (5): Let $0 \neq f \in I'(x,y)$, $0 \neq g \in I'(x,z)$, $0 \neq h \in I'(z,y)$, $(f-h)(x-z,y) = 0 = (f-g)(x,y-z)$ and $z \neq x+y$, i.e., the negation of (5), then $f \in I'(x,y) \cap I'(z,y) \cap I'(x,z)$ by Lemma 2 and 3, and $z \neq x+y$. Thus (4) does not hold.

(5) ⇒ (4): If $f \in I'(x,y) \cap I'(x,z) \cap I'(z,y)$ and suppose that $z \neq x+y$, then $0 = (f-f)(x-z,y) > 0$ by (5) yielding a contradiction, and the proof of the theorem is complete.

REMARK. (2) in Theorem 4 appeared in ([4] Theorem 1.2) except that the domain of the duality mapping I has been changed. The change is unnecessary.

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