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ONE-DIMENSIONAL GAME OF LIFE AND ITS GROWTH FUNCTIONS

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ABSTRACT. We start with finitely many 1's and possibly some 0's in between. Then each entry in the other rows is obtained from the Base 2 sum of the two numbers diagonally above it in the preceding row. We may formulate the game as follows:

Define $d_{i,j}$ recursively for i, a non-negative integer, and j an arbitrary integer by the rules:

$$d_{0,j} = \begin{cases} 1 & \text{for } j = 0,k \\ 0 \text{ or } 1 & \text{for } 0 < j < k \end{cases} \tag{I}$$

$$d_{0,j} = 0 \text{ for } j < 0 \text{ or } j > k \tag{II}$$

$$d_{i+1,j} = d_{i,\ j+1} \pmod 2 \text{ for } i \geq 0. \tag{III}$$

Now, if we interpret the number of 1's in row i as the coefficient a_i of a formal power series, then we obtain a growth function,

$f(x) = \sum_{i=0}^{\infty} a_i x^i$. It is interesting that there are cases for which this growth function factors into an infinite product of polynomials. Furthermore, we shall show that this power series never represents a rational function.

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1. INTRODUCTION

To explain in words:

(I) describes the starting configuration with finitely many 1's, and possibly some 0's in between. (Picture this as row zero).

(II) says that all entries on both sides of the starting configuration are zero. (Note that considering the 0's on both sides contributes nothing).

(III) says that each entry (in the other rows) is obtained from the Base 2 sum of the two numbers diagonally above it in the preceding row.

Now, if we interpret the number of 1's in row i as the coefficient a_i of a formal power series, then we obtain a growth function:

$f(x) = \sum_{i=0}^{\infty} a_i x^i$. It is interesting that there are cases for which this growth function factors into an infinite product of polynomials. But there are cases in which the pattern is less regular. Nevertheless, we shall show the following.

THEOREM. In a One-dimensional Game of Life, no matter what the starting configuration is, (finitely many 1's and possibly some 0's in between), the associated growth function never represents a rational function.

First we look at some examples:

EXAMPLE 1. Suppose the starting configuration has one 1. We get the following configuration called "Fundamental Configuration" throughout this paper.

Row	Term	
0	1	1
1	$2x$	1 0 1
2	$2x^2$	1 0 0 0 1
3	$4x^3$	1 0 1 0 1 0 1
4	$2x^4$	1 0 0 0 0 0 0 0 1
5	$4x^5$	1 0 1 0 0 0 0 0 1 0 1
6	$4x^6$	1 0 0 0 1 0 0 0 1 0 0 0 1
.	.	. . . .

The resulting growth function is:

$$f(x) = \sum_{i=0}^{\infty} a_i x^i = 1 + 2x + 2x^2 + 4x^3 + 2x^4 + 4x^5 + 4x^6 + . . . \quad .$$

We observe that the configuration above is essentially the configuration obtained by reducing modulo 2 each element of Pascal's triangle of binomial coefficients. To obtain the reduced Pascal's triangle from configuration in example 1 simply remove every second entry of each row beginning with the entry 0 following the initial 1 of that row. Each entry removed will be a zero.

```
      1                     1
    1 0 1     ---->        1 1
  1 0 0 0 1              1  0  1
 .............         .............
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This connection with Pascal's triangle can be used to explain the fact that, for each $j \geq 1$, row 2^j has precisely two non-zero entries, one at each end. In other words, for every integer $j \geq 1$, row 2^j has two copies of starting configuration, one at each end. Therefore, $a_{2^j} = 2a_0$.

Worthy of notice is that Wolfram [3] has shown: In Pascal's Triangle of binomial coefficients reduced modulo 2, when infinitely many rows are included, the limiting pattern exhibits a fractal self-similarity and is characterized by a "fractal dimension" $\log_2 3$.

DEFINITION 1. For any integer $j \geq 1$, the set of rows between row 2^{j-1} and $2^j - 1$ inclusive, is called the jth Stage. Note that row 2^j, $j \geq 1$, is the initial row of the jth Stage. Stage 0, is the 0th row or the initial row of the configuration.

EXAMPLE 2. If $d_{0,j} = 1$, $j = 0,1$ then we get the following configuration:

Row	Term		
0	2	1 1	<-- stage 0
1	$4x$	1 1 1 1	<-- stage 1
2	$4x^2$	1 1 0 0 1 1	
3	$8x^3$	1 1 1 1 1 1 1 1	<-- stage 2
4	$4x^4$	1 1 0 0 0 0 0 0 1 1	
5	$8x^5$	1 1 1 1 0 0 0 0 1 1 1 1	
6	$8x^6$	1 1 0 0 1 1 0 0 1 1 0 0 1 1	
7	$16x^7$	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	
8	$4x^8$	1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1	
.	.	. . . .	

The resulting growth function $g(x)$ may be written as:

$$g(x) = 2 + 4x + 4x^2 + 8x^3 + 4x^4 + 8x^5 + 8x^6 + 16x^7 + 4x^8 + \dots$$

EXAMPLE 3. Let $d_{0,j} = 1$ for $j = 0,1,2$. This yields the following configuration:

Row	Term	
0	3	1 1 1
1	$4x$	1 1 0 1 1
2	$6x^2$	1 1 1 0 1 1 1
3	$6x^3$	1 1 0 1 0 1 0 1 1
4	$6x^4$	1 1 1 0 0 0 0 0 1 1 1
5	$8x^5$	1 1 0 1 1 0 0 0 1 1 0 1 1
6	$12x^6$	1 1 1 0 1 1 1 0 1 1 1 0 1 1 1
7	$10x^7$	1 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 1
8	$6x^8$	1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1
.	.	. . . .
.	.	. . . .
.	.	. . . .

The growth function $h(x)$ is:

$$h(x) = 3 + 4x + 6x^2 + 6x^3 + 6x^4 + 8x^5 + 12x^6 + 10x^7 + 6x^8 + \dots$$

FACTORIZATION OF THE FUNDAMENTAL GROWTH FUNCTION. If we look at the configurations in examples 1 - 3 carefully, we observe that there is in each a recurrent triangular pattern. This is dramatically illustrated in the more extensive computer printouts of figures 1 - 4, reproduced below. Note that in these figures the zeros are not printed.

In fundamental configuration (Fig. 1) since in each Stage we have two copies of the preceding triangle (or Stage), hence we may factor its growth function, $f(x)$, as follows:

$$f(x) = (1 + 2x)(1 + 2x^2)(1 + 2x^4)(1 + 2x^8)(1 + 2x^{12}) \dots$$

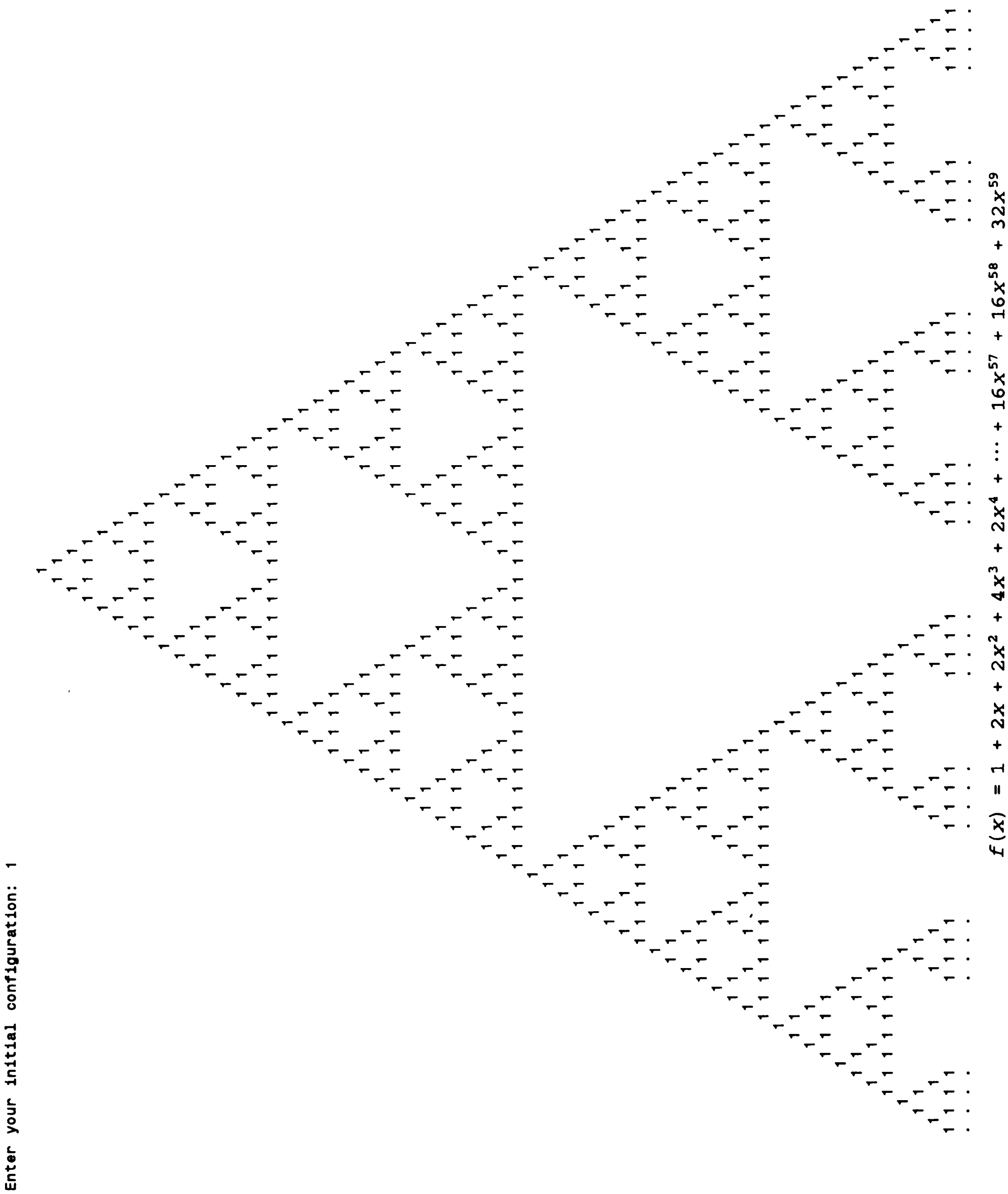

Fig. 1. The starting configuration has only one 1.

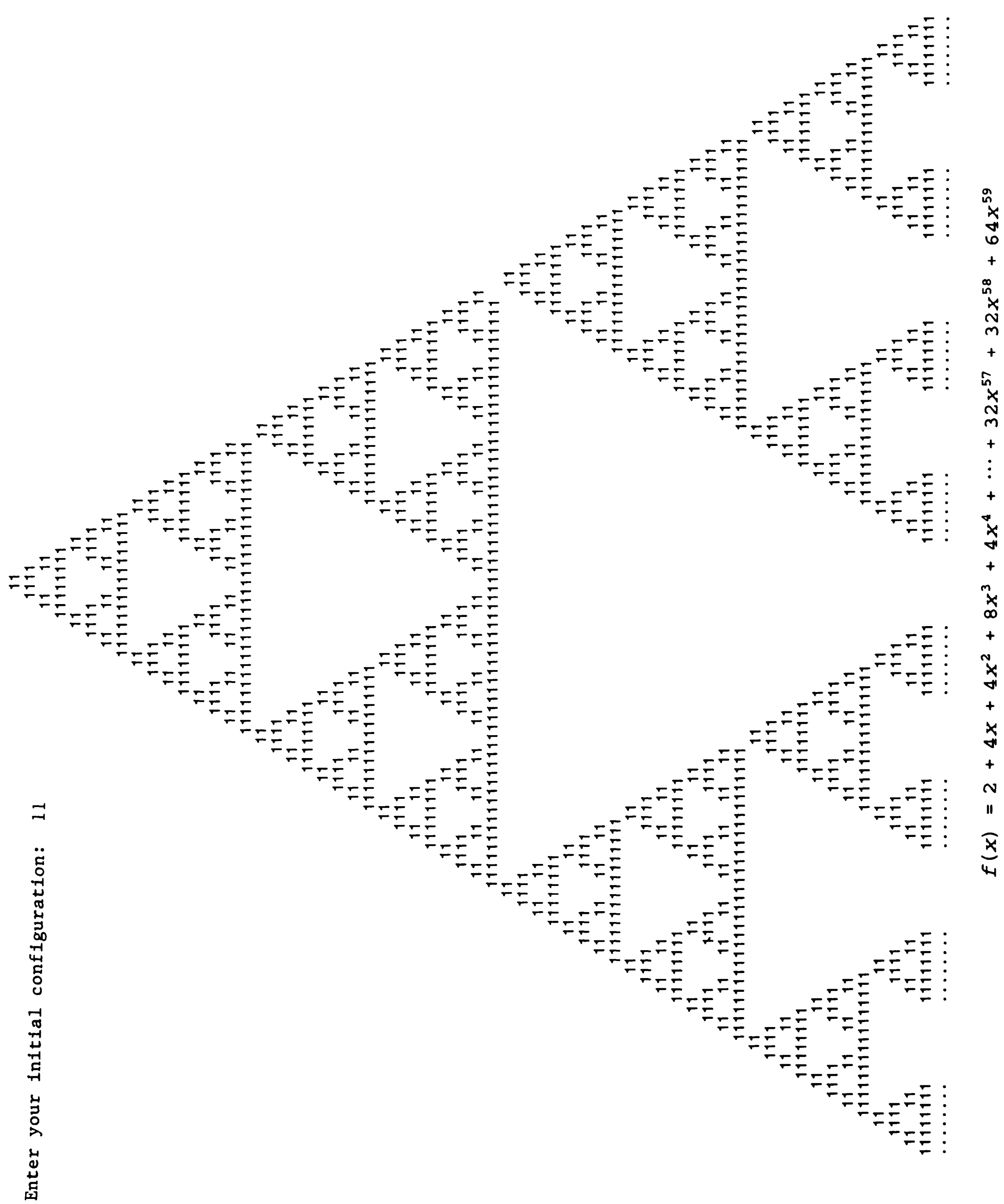

Fig. 2 The starting configuration is "11", that is, two 1's

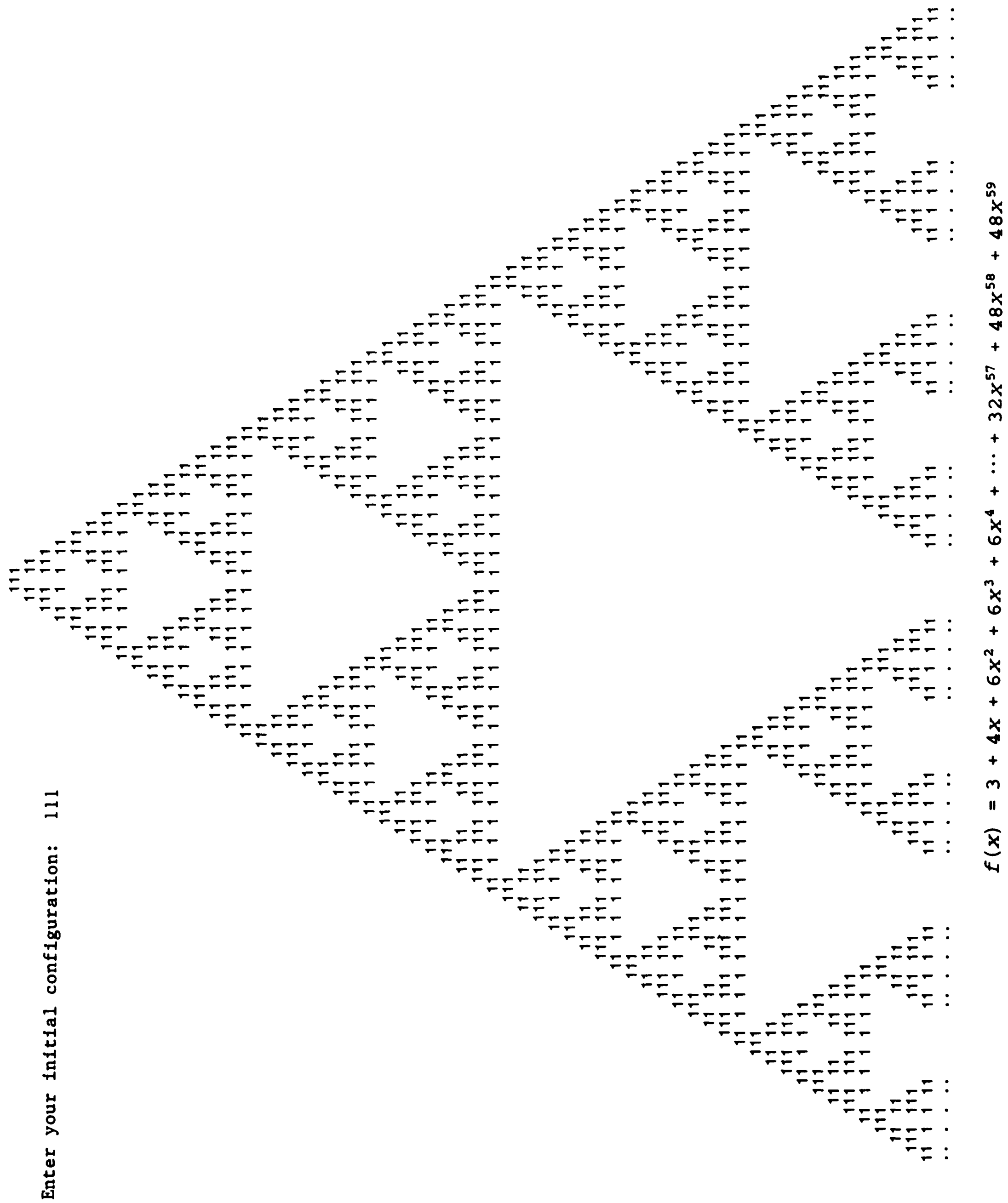

Fig. 3 The starting configuration is "111, that is, three 1's"

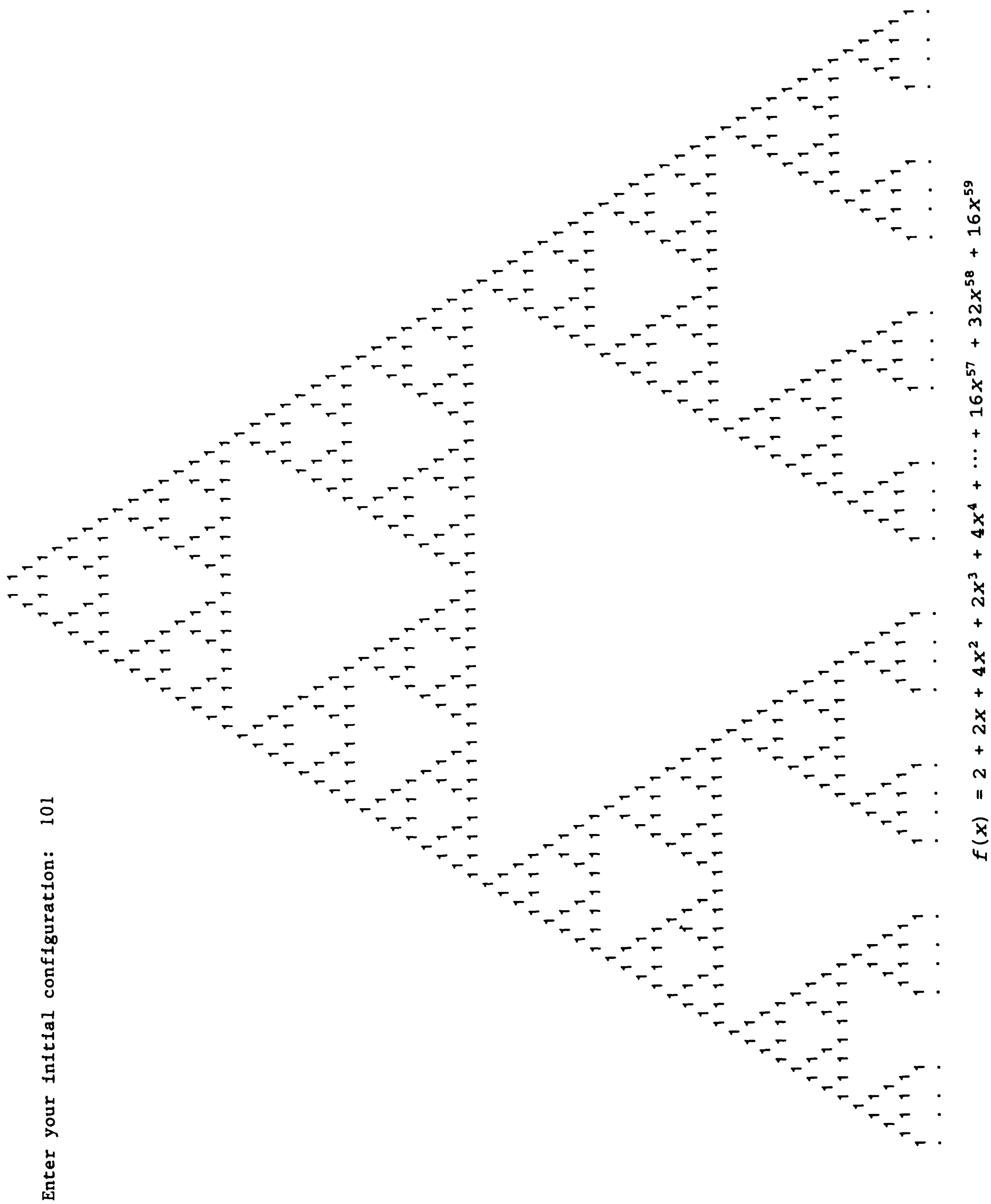

Fig. 4 The starting Configuration is "101, that is, one-zero-one"

Because in Fig. 1, the triangle formed of stages 0 and 1 gives the first factor $1 + 2x$. Then since it duplicates in the next stage (i.e., in Stage 3), and the corresponding term involving x of the initial row of stage 3 is x^2 (while considering the first triangle as a whole, i,e., equivalent to 1), then this yields to $1 + 2x^2$, as the second factor of f(x) in the factorized form, etc.

Similarly, the growth function in example 2 (see Fig. 2) can be factored as:

$$g(x) = (2 + 4x)(1 + 2x^2)(1 + 2x^4)(1 + 2x^8)(1 + 2x^{16}) \dots$$

Note that the factorization of f and g exhibit infinitely many zeros of the power series which cluster at every point inside the unit circle in complex plane. So f and g must be non-rational.

In example 3, the pattern is less regular. The "irregularity" first appears in the last row of stage 0, and therefore we can not factor the growth function in this way. Nevertheless, we shall show that the associated growth function of a general configuration is always non-rational.

It is obvious that there is a natural relationship between an arbitrary starting configuration and the "fundamental configuration" since a general configuration is simply a "mode 2" sum of shifts of the fundamental one. Indeed it follows easily from the definition given below.

DEFINITION. If the starting configuration has k number of 1's and l number of 0's interspersed among them at the columns $j_1, j_2, \dots, j_l$ then its entries, $c_{i,j}$, can be obtained by the formula:

$$c_{i,j} = \sum_{\substack{t=0 \\ t \neq j_1, \dots, j_l}}^{k+l} d_{i,j-t} \pmod 2$$

where k and l are integers with $k \geq 1$ and $l \geq 0$.

Now, this definition including the fact that new stages on fundamental configuration begin at the rows 2^j imply, no matter what the starting configuration is, the new stages always start at the rows $n = 2^j$; and the corresponding terms of the growth series are $a_n x^{2^j}$, where $a_n = 2a_0$.

THEOREM. In a One-dimensional Game of Life, no matter what the starting configuration is, (finitely many 1's and possibly some 0's in between), the associated growth function is non-rational.

PROOF. Suppose, on the contrary, there is some case in which the associated growth function

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$$

is rational. It is shown in [2] that if a power series with integral coefficients represents a rational function, then it can be expressed in the form $\frac{P(x)}{Q(x)}$ where P(x), Q(x) are polynomials with integral coefficients and $Q(0) = 1$.

Therefore, if f(x) is rational then there are polynomials $P(x) = \sum_{i=0}^{n} c_i x^i$, and $Q(x) = \sum_{i=0}^{m} b_i x^i$, where c_i, $b_i \in Z$ and $Q(0) = 1$ (i.e., $b_0 = 1$) such that

$f(x) = \frac{P(x)}{Q(x)}$.

By doing simple algebra, this in turn implies that:

$$a_i = c_i - (b_1a_{i-1} + b_2a_{i-2} + \ldots + b_ma_{i-m}) \text{ for } i \leq n \text{ and,} \quad (1)$$

$$a_i = -(b_1a_{i-1} + b_2a_{i-2} + \ldots + b_ma_{i-m}) \quad \text{for } i > n. \quad (2)$$

(Set $a_i = 0$ for $i < 0$.) That is if f is rational then there is a linear recurrence relation (2) for coefficients of the power series after a certain number of terms. Assuming that such a linear recursion exists, there are integers $y_1, \ldots, y_m$ such that:

$$a_i = y_1a_{i-1} + y_2a_{i-2} + \ldots + y_ma_{i-m}, \text{ for } i > n \ (a_i = 0, \text{ for } i < 0). \quad (3)$$

Now, we may choose i large enough so that $i = q = 2^j$, for some j and $q > m$. Note that row q represents the first row of a new stage. So, we have:

$$a_q = 2a_0, \ a_{q+1} = 2a_1, \ \ldots, \ a_{q+m-1} = 2a_{m-1}, \ a_{q+m} = 2a_m. \quad (4)$$

From (3) and (4) we have:

$$\begin{aligned} a_{q+m+1} &= y_1a_{q+m} + y_2q_{q+m-1} + \ldots + y_ma_{q+1} && \text{By (3)} \\ &= 2a_my_1 + 2a_{m-1}y_2 + \ldots + 2a_1y_m && \text{By (4)} \\ &= 2((y_1a_m + y_2a_{m-1} + \ldots + y_ma_1)4 \\ &= 2a_{m+1} && \text{(By setting } i = m+1 \text{ in (3))} \end{aligned}$$

Similarly

$$\begin{aligned} a_{q+m+2} &= y_1a_{q+m+1} + y_2a_{q+m} + \ldots y_ma_{q+2} \\ &= 2a_{m+1}y_1 + 2a_my_2 + \ldots + 2a_2y_m \\ &= 2(y_1a_{m+1} + y_2a_m + \ldots + y_ma_2) \\ &= 2a_{m+2} \end{aligned}$$

.

.

.

$$\begin{aligned} a_{q+(q-1)} &= y_1a_{2q-2} + y_2a_{2q-3} + \ldots + y_ma_{2q-m-1} \\ &= 2a_{q-2}y_1 + 2a_{q-3}y_2 + \ldots + 2a_{q-m-1}y_m \\ &= 2a_{q-1}. \end{aligned}$$

Also:

$$\begin{aligned} a_{2q} &= y_1a_{2q-1} + y_2a_{2q-2} + \ldots + y_mq_{2q-m} \\ &= y_1(2a_{q-1}) + y_2(2a_{q-2}) + \ldots + y_m(2a_{q-m}) \\ &= 2a_q \\ &= 2(2a_0) && \text{(By 4)} \\ &= 4a_0 . \end{aligned}$$

Therefore, $a_{2q} = 4a_0$. But this is a contradiction, because row $2q = 2(2^j) = 2^{j+1}$ represents the first row of a new stage, i.e. Stage $(j + 1)$, and hence $a_{2q} = 2a_0$ and this completes the proof of the theorem. □

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