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AN APPLICATION OF KKM-MAP PRINCIPLE

A. CARBONE

Dipartimento di Matematica
Universitá degli studi della Calabria
87036 Arcavacata di Rende (Cosenza) - ITALY

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ABSTRACT. The following theorem is proved and several fixed point theorems and coincidence theorems are derived as corollaries. Let C be a nonempty convex subset of a normed linear space X, $f: C \to X$ a continuous function, $g: C \to C$ continuous, onto and almost quasi-convex. Assume that C has a nonempty compact convex subset D such that the set

$$A = \{y \in C: \| g(x) - f(y) \| \geq \| g(y) - f(y) \| \text{ for all } x \in D\}$$

is compact.

Then there is a point $y_0 \in C$ such that $\| g(y_0) - f(y_0) \| = d(f(y_0), C)$.

KEY WORDS AND PHRASES. Almost quasi-convex functions, fixed points, coincidence points.
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1. INTRODUCTION.

There has been given a variety of applications of KKM-map principle by Ky Fan [1] in areas like fixed point theory, approximation theory, minimax theory, potential theory and variational problems. For further applications we refer to [2].

Recently Prolla [4] proved the following result using fixed point theorems for multivalued mappings. In this note we extend his theorem and our proof will follow KKM-map principle.

Let C be a compact, convex subset of a Banach space $X, f: C \to X$ a continuous function and $g: C \to C$ a continuous, almost affine and onto map. Then there is a $y_0 \in C$ such that

$$\| g(y_0) - f(y_0) \| = d(f(y_0), C).$$

Recall that a map $g: C \to X$ is almost affine if

$$\| g(\lambda x_1 + (1-\lambda)x_2) - y \| \leq \lambda \| g(x_1) - y \| + (1-\lambda) \| g(x_2) - y \|$$

for all $x_1, x_2 \in C$ and $y \in X$.

Clearly a linear map is almost affine, but not conversely.

We have taken an almost quasi-convex map g.

DEFINITION. A map $g: C \to X$ is said to be almost quasi-convex if, for every $t \in X$ and $r > 0$, the set $\{u \in C: \| g(u) - t \| < r\}$ is convex.

An almost quasi-convex condition is more general than almost affine condition.

We use the following well-known result (Lin [3]) to derive our theorem given below.

THEOREM 1.1. Let C be a nonempty convex subset of a topological vector space. Let $B \subset C \times C$ be such that

i) for each $x \in C$, the set $\{y \in C: (x,y) \in B\}$ is closed in C;

ii) for each $y \in C$ the set $\{x \in C: (x,y) \notin B\}$ is empty or convex;

iii) $(x,x) \in B$ for each $x \in C$; and

iv) C has a nonempty compact convex subset D such that the set

$$A = \{y \in C: (x,y) \in B \text{ for all } x \in D\}$$

is compact.

Then there exists a point $y_0 \in C$ such that $C \times \{y_0\} \subseteq B$.

2. MAIN RESULTS.

Now we prove our results.

THEOREM 2.1. Let C be a nonempty convex subset of a normed linear space X, $f: C \to X$ a continuous function, $g: C \to C$ continuous, onto and almost quasi-convex function. (*) Assume that C has a nonempty compact convex subset D such that the set

$$A = \{y \in C: \| g(x) - f(y) \| \geq \| g(y) - f(y) \| \text{ for all } x \in D\}$$

is compact.

Then there is a point $y_0 \in C$ such that $\| g(y_0) - f(y_0) \| = d(f(y_0), C)$.

PROOF. Set

$$B = \{(x,y) \in C \times C: \| g(x) - f(y) \| \geq \| g(y) - f(y) \| \}.$$

Then the set $\{y \in C: (x,y) \in B\}$ is closed in C since f and g are continuous. It is easy to see that $(x,x) \in B$ for each $x \in C$.

We have to show that the set

$$M = \{x \in C: (x,y) \notin B\} = \{x \in C: \| g(x) - f(y) \| < \| g(y) - f(y) \| \}$$

is convex or empty.

Since g is an almost quasi-convex function, therefore M is convex.

By Theorem 1.1 we get that there is a point $y_0 \in C$ such that

$$\| g(y_0) - f(y_0) \| = d(f(y_0), C).$$

In case the convex set C is compact we may take $C = D$.

NOTE. Condition (*) is equivalent to the following.

Let D be a nonempty compact convex subset of C, K be a nonempty compact subset of C such that for each $y \in C \backslash K$ there exists an $x_0 \in D$ such that

$$\| g(x_0) - f(y) \| < \| g(y) - f(y) \|.$$

If $C = K = D$ and g is almost affine then we get Prolla's result stated below. Let C be a compact convex subset of a normed linear space X and $f: C \to X$ a continuous function. Let $g: C \to X$ be a continuous, onto and almost affine map. Then there exists a $y_0 \in C$ such that

$$\| g(y_0) - f(y_0) \| = d(f(y_0), C).$$

NOTE. (i) If $f(y_0) \in C$ then we get a coincidence result; and (ii) If $g = I$, an identity function, then the above result is a well-known theorem due to by Ky Fan [1]. This theorem has interesting applications in fixed point theory, approximation theory and variational problems. We give a sample application in fixed point theory.

EXAMPLE: Let C be a compact convex subset of a normed linear space X and $f: C \to X$ a continuous map. If $f(x) \neq x$, then assume that the line segment $[x, f(x)]$ has at least two elements of C. Then f has a fixed point.

By taking $g = I$, we get there is a $y_0 \in C$ such that

$$\| y_0 - f(y_0) \| = d(f(y_0), C).$$

Now, if $y_0 \neq f(y_0)$ then there is a $z \in C$ such that

$$z = \lambda f(y_0) + (1-\lambda) y_0, \quad 0 < \lambda < 1$$

and

$$\begin{aligned} \| y_0 - f(y_0) \| \leq \| z - f(y_0) \| &= \| \lambda f(y_0) + (1-\lambda) y_0 - f(y_0) \| \\ &= (1-\lambda) \| f(y_0) - y_0 \| < \| f(y_0) - y_0 \| \end{aligned}$$

a contradiction, so $y_0 = f(y_0)$.

We could derive several other interesting results on fixed point theorems as corollaries.

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