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UNORDERED LOVE IN INFINITE DIRECTED GRAPHS

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ABSTRACT. A digraph $D=(V,A)$ has the Unordered Love Property (ULP) if any two different vertices have a unique common outneighbor. If both (V,A) and (V,A^{-1}) have the ULP, we say that D has the SDULP.

A love-master in D is a vertex v_0 connected both ways to every other vertex, such that $D-v_0$ is a disjoint union of directed cycles.

The following results, more or less well-known for finite digraphs, are proven here for D infinite: (i) if D is loopless and has the SDULP, then either D has a love-master, or D is associable with a projective plane, obtainable by taking V as the set of points and the sets of outneighbors of vertices as the lines; (ii) every projective plane arises from a digraph with the SDULP, in this way.

KEY WORDS AND PHRASES. Digraph, projective plane, friendship graph.
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1. INTRODUCTION.

Throughout, $D=(V,A), A\subseteq V\times V$, will be a directed graph. Loosely following the terminology facetiously introduced by Hammersley [4], we will say that D has the Unordered Love Property (abbreviated ULP) if $u,v\in V$ and $u\neq v$ imply the existence of a unique $w=w(u,v)$ such that $(u,w),(v,w)\in A$. Hammersley mainly considers the unfortunately named Self-Dual Unordered Love Property (SDULP), in which both (V,A) and (V,A^{-1}) have the ULP.

Both properties are analogues of the friendship property in undirected graphs, the finite possessors of which are the famous friendship graphs [10]. The reason for the word "unordered" is the role of unordered pairs in the definition. There is also an Ordered Love Property defined by Hammersley, in which for each ordered pair $(u,v)\in V\times V$, if $u\neq v$ then there is a unique $w\in V$ such that $(u,w),(w,v)\in A$. We shall not consider this property here, but remark that it should not be confused with the more exigent requirement satisfied by Knuth's [6] digraph realizations of the central groupoids of Evans [3], in which the requirement that $u\neq v$ is omitted from the definition. Thus Knuth's digraphs are among Hammersley's (with the OLP), and it turns out that the containment is proper. So far as we know, there has not yet been any investigation beyond Knuth's of the liaison between digraphs with the OLP, ULP, or SDULP, and algebraic structures like the central groupoids.

Hammersley [4] makes the connection between finite digraphs with the SDULP and projective planes (or degenerate projective planes). However, perhaps because he is greatly concerned with loops, he misses the fact that in loopless finite digraphs, the ULP implies the SDULP. This, and the conclusion of Theorem 1, below, for finite loopless digraphs with the ULP, follow easily from the famous theorem of de Bruijn and Erdös [2] or its improvement by Ryser [8]. In the infinite case, a moment's thought shows that looplessness and the ULP do not imply the SDULP, nor the conclusion of Theorem 1. The loopless infinite digraphs with the ULP do not form such a neat package; this paper is about such graphs with the SDULP, which do.

The analogue of Theorem 2, below, for the finite projective planes follows easily from the existence of perfect matchings in finite regular bipartite graphs (see [1], Chapter 8); by such a matching we match the points and lines of a finite projective plane so that the matched points and lines are not incident, and then proceed as in the proof of Theorem 2.

2. DEFINITIONS.

Suppose that $D = (V, A)$ is a directed graph. For $v \in V$,

$In(v) \; = \{u \in V; (u,v) \in A\}$,

$Out(v) = \{u \in V; (v,u) \in A\}$, and, following [1],

$id(v) \; = |In(v)|$, the cardinality of $In(v)$, and

$od(v) \; = |Out(v)|$

If D has the ULP, let $\theta(u,v)$ denote the unique member of $Out(u) \cap Out(v)$, for $u, v \in V, u \neq v$.

A vertex $v_0 \in V$ is a love-master in G if $(u, v_0), (v_0, u) \in A$ for each $u \in V \backslash \{v_0\}$, and $D - v_0$ is a disjoint union of directed cycles of length ≥ 2. In case V is infinite, we allow as a directed cycle any infinite path isomorphic to the digraph with integers as vertices, and directed edges $(n, n+1)$, for each integer n.

3. MAIN RESULTS.

THEOREM 1. Suppose that $D = (V, A)$ is a loopless infinite digraph with the SDULP. Then either

(a) some $v_0 \in V$ is a love-master in D or

(b) if we take $\mathcal{P} = V, \mathcal{L} = \{Out(v); v \in V\}$ and $\in$ as the incidence relation between points and lines, then $(\mathcal{P}, \mathcal{L}, \in)$ is a projective plane.

THEOREM 2. Suppose that $Z = (\mathcal{P}, \mathcal{L}, \in)$ is an infinite projective plane. Then there is an infinite loopless digraph $D = (V, A)$ with the SDULP such that the projective plane associated with D as in Theorem 1(b) is isomorphic to Z.

4. PROOFS.

LEMMA 1. Suppose that $D = (V, A)$ is a digraph with the ULP, $u, v \in V$, and $(u, v) \notin A$. Then $id(v) \leq od(u)$.

PROOF. The map $f: In(v) \to Out(u)$ defined by $f(w) = \theta(u, w)$ is well defined, since $u \notin In(v)$. If $w_1, w_2 \in In(v)$ and $w_1 \neq w_2$, then $\{v\} = Out(w_1) \cap Out(w_2)$. Since $f(w_i) \in Out(w_i), i = 1,2$, and $v \notin Out(u)$, it follows that $f(w_1) \neq f(w_2)$. Thus f is an injection of $In(v)$ into $Out(u)$.

COROLLARY 1. If $D = (V, A)$ is a loopless digraph with the SDULP, then $id(v) = od(v)$ for each $v \in V$.

PROOF. Take $u = v$ in Lemma 1.

LEMMA 2. Suppose that $D = (V, A)$ is a loopless infinite digraph with the SDULP; then $v_0 \in V$ is a love-master in D if and only if $In(v_0) = V \backslash \{v_0\}$.

PROOF. The "only if" assertion is trivial.

We shall see that $In(v_0) = V\backslash\{v_0\}$ implies that $Out(v_0) = V\backslash\{v_0\}$. Suppose that $v \in V\backslash\{v_0\}$. Since (V, A^{-1}) has the ULP, $In(v_0) \cap In(v) = \{u\}$, for some $u \notin \{v, v_0\}$ and $In(v) \cap In(u) = \{w\}$, for some $w \notin \{u, v\}$. If $w \neq v_0$ then $w \in V\backslash\{v_0\} = In(v_0)$, which implies that $w \in In(v_0) \cap In(v) = \{u\}$, whence $w = u$, an absurdity. Therefore, $v_0 = w \in In(v)$, so $v \in Out(v_0)$. Since $v \in V\backslash\{v_0\}$ was arbitrary, it follows that $Out(v_0) = V\backslash\{v_0\}$.

It remains to be seen that $D - v_0$ is the disjoint union of directed cycles, of length ≥ 2 (including, possibly, two-sidedly infinite directed paths). Observe that, for $v \in V\backslash\{v_0\}$, $Out(v) = \{v_0, \theta(v, v_0)\}$, since v has exactly one common outneighbor with v_0 and $Out(v_0) = V\backslash\{v_0\}$. Similarly, $In(v)$ has exactly one member beside v_0. Thus $id_{D-v_0}(v) = od_{D-v_0}(v) = 1$. The remainder of the proof is easy.

LEMMA 3. Suppose that $D = (V, A)$ is a loopless infinite digraph with the SDULP, and $u \in V$ has finite outdegree or indegree. Then D has a love-master.

PROOF. By Corollary 1, $id(u) = od(u) < \infty$. Since each $v \in V\backslash\{u\}$ has a common outneighbor with u, and V is infinite, $Out(u)$ finite, some $v_0 \in Out(u)$ must have infinite indegree. By Lemma 1, $w \in In(v_0)$ for every $w \in V$ with $id(w) = od(w)$ finite. By Lemma 2, it suffices to show that v_0 is the only vertex of infinite indegree.

If $v_1 \neq v_0$ is another vertex of infinite indegree, then, by Lemma 1 again, $w \in In(v_1)$ whenever $id(w) = od(w) < \infty$. Thus there can be at most one such w, since $|In(v_0) \cap In(v_1)| = 1$. But there are infinitely many such w, since, by Lemma 1 again, if $w \in V\backslash Out(u)$, then $id(w) \leq od(u) < \infty$. This contradiction shows that no such v_1 exists, so v_0 is a love-master.

LEMMA 4. Suppose a geometry $Z = (\mathcal{P}, \mathcal{L}, \in)$ satisfies

(i) any two distinct points are incident with exactly one line,

(ii) any two distinct lines are incident with exactly one point, and

(iii) each line is incident to at least three points, and there are at least two lines.

Then Z is a projective plane.

The proof is straightforward, and is omitted.

PROOF OF THEOREM 1. By Lemma 3, we may as well assume that $id(u) = od(u)$ is infinite for each $u \in V$. Let $Z = (\mathcal{P}, \mathcal{L}, \in)$ be as in (b). Z satisfies condition (i) of Lemma 4 because (V, A^{-1}) has the ULP, condition (ii) because (V, A) has the ULP, and condition (iii) because $od(u)$ is infinite for each $u \in V$. Thus Z is a projective plane.

LEMMA 5. Every infinite regular bipartite graph with infinite degree admits a perfect matching.

PROOF. This is an easy application of a theorem of König [7, p. 221]. See also [9, p. 142].

PROOF OF THEOREM 2. We form a bipartite graph G with bipartition $\mathcal{P}, \mathcal{L}$ by defining $PL \in E(G)$ if and only if $P \notin L$ (i.e., the point P and the line L are *not* incident). G clearly satisfies the hypothesis of Lemma 5, and so admits a perfect matching. Therefore, there is a bijection $g: \mathcal{P} \to \mathcal{L}$ such that $P \notin g(P)$ for each $P \in \mathcal{P}$. Define $D = (V, A)$ be setting $V = \mathcal{P}$ and $(P, Q) \in A$ if and only if $Q \in g(P)$. The condition $P \notin g(P)$ implies that D has no loops. The bijectivity of g and the two-points-determine-a-line, two-lines-determine-a-point properties of Z imply, respectively, that (V, A^{-1}) and (V, A) have the ULP. If each line of Z is identified with the set of points on it, then the lines of Z are the same as the lines of the projective plane associated with D; the points of the two planes are already the same, and the incidence relations become identical.

5. REMARKS AND PROBLEMS.

A digraph $D = (V, A)$ is *anti-symmetric*, or *oriented*, if and only if $(u, v) \in A$ implies that $(v, u) \notin A$, for all $u, v \in V$. Note that an oriented digraph is necessarily loopless.

PROBLEM 1. Is every projective plane associated, as in Theorem 1(b) and Theorem 2, with an oriented digraph with the SDULP?

The analogue of Theorem 2 for finite projective planes says that every finite projective plane has an incidence matrix B with $tr(B) = 0$. If the answer to Problem 1's question is yes, then every finite projective plane has an incidence matrix B satisfying $tr(B^2) = 0$.

It is clear that if $D = (V, A)$ has a love-master, then D is isomorphic, as a digraph, to (V, A^{-1}). When (V, A) has the SDULP and is associated with a projective plane Z, then (V, A^{-1}) is associated with a plane isomorphic to the dual Z^* of Z. ["In" is a bijection from the lines of Z^* to the lines of the plane associated with (V, A^{-1}), and "Out" is a bijection from the points of the (V, A^{-1}) plane to the points of Z^*; the fact that $u \in In(v)$ iff $v \in Out(u)$ says that these correspondences preserve incidence.] Since there are projective planes, both finite and infinite, which are not isomorphic to their duals (see [5]), it follows from Theorem 2 that there exist both finite and infinite loopless digraphs (V, A) with the SDULP which are not isomorphic to (V, A^{-1}).

PROBLEM 2. Are there any projective planes Z for which any digraph $D = (V, A)$ associated with Z as in Theorem 2 is isomorphic to (V, A^{-1})?

PROBLEM 3. Are there any projective planes for which there is only one loopless digraph, up to isomorphism, associated with the plane, as in Theorem 2?

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