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ON A CONDITIONAL CAUCHY FUNCTIONAL EQUATION OF SEVERAL VARIABLES AND A CHARACTERIZATION OF MULTIVARIATE STABLE DISTRIBUTIONS

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ABSTRACT. The general solution of a conditional Cauchy functional equation of several variables is obtained and its applications to the characterizations of multivariate stable distributions are studied.

KEY WORDS AND PHRASES. Multivariate stable distribution, characteristic function, canonical representation, independent and identically distributed, characterization.
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1. INTRODUCTION.

The purpose of this note is to solve the conditional Cauchy functional equation of several variables

$$f(ax) = a^{\alpha} f(x), \ \forall x \in R^n, \tag{1.1}$$

for a continuous function f on R^n, where $\alpha > 0$ and $a = 2,3$.

We apply solutions of the equation (1.1) to problems of characterization of multivariate stable distributions, and generalize a result of Eaton [1] on characterization of univariate stable distributions based on a functional equation.

2. MULTIVARIATE STABLE DISTRIBUTIONS

A random variable with distribution function F is said to have univariate stable distribution if for every $b_1, b_2 > 0$, $c_1, c_2 \in R$, there correspond a $b > 0$ and $c \in R$ such that

$$F\left(\frac{y-c_1}{b_1}\right) * F\left(\frac{y-c_2}{b_2}\right) = F\left(\frac{y-c}{b}\right), \ \forall y \in R,$$

where $*$ denotes the convolution operator of distribution functions. Lévy [4] showed that a univariate stable distribution has characteristic function $\phi(t)$ of the form

$$\ln \phi(t) = i\mu t - \gamma |t|^{\alpha}\left[1 + i\beta \frac{t}{|t|} w(|t|, \alpha)\right], \ \forall t \in R, \tag{2.1}$$

where the constants γ, β, α satisfy the conditions $\gamma \geq 0$, $|\beta| \leq 1$, $0 < \alpha \leq 2$, and where

μ is a real number. The function $w(|t|,\alpha)$ is given by

$$w(|t|,\alpha) = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right), & \text{if } \alpha \neq 1, \\ \frac{2}{\pi} \ln|t|, & \text{if } \alpha = 1. \end{cases}$$

Multivariate generalizations of univariate stable distributions were introduced by Lévy [5] and Feldheim [2]. Let x denote an $n \times 1$ vector. A multivariate distribution $G(x)$ is said to be multivariate stable if to every pair of scalars $b_1, b_2 > 0$, and real vectors $c_1, c_2 \in R^n$, there corresponds a scalar $b > 0$ and a real vector $c \in R^n$ such that

$$G\left(\frac{x-c_1}{b_1}\right) * G\left(\frac{x-c_2}{b_2}\right) = G\left(\frac{x-c}{b}\right), \quad \forall x \in R^n.$$

It has been shown that the characteristic function $\phi(t)$ of a multivariate stable distribution has canonical representation

$$\ln \phi(t) = i\mu' t - (t't)^{\frac{\alpha}{2}} \left[g_1\left(\frac{t}{\|t\|}\right) + i g_2\left(\frac{t}{\|t\|}\right) \right], \quad \forall t \in R^n, \tag{2.2}$$

where $0 < \alpha \leq 2$, $\mu \in R^n$, g_1 and g_2 are continuous functions on R^n given by integral representations (Lévy [5]). We remark that the explicit algebraic representations of g_1 and g_2 for multivariate stable distributions obtained by Press [7] is not for all multivariate distributions. It is complemented by examples in Paulauskas [6], indicating that the closed expressions of g_1 and g_2, analogous to univariate case, are still unknown.

3. THE SOLUTIONS OF A CONDITIONAL CAUCHY FUNCTIONAL EQUATION OF SEVERAL VARIABLES.

In this section, we will derive the general solution f of the Cauchy-type functional equation (1.1), where $\alpha > 0$ is given, and $a = 2,3$.

Let $S = \{s \in R^n \mid \|s\| = 1\}$. For each $s \in S$, define $G_s = \{rs \mid r > 0\}$. Then R^n is partitioned into disjoint union of G_s, $s \in S$, and $\{0\}$, i.e.

$$R^n = \left(\bigcup_{s\in S} G_s\right) \cup \{0\}.$$

Let f: $R^n \to R$ be continuous solution of the equation (1.1). On each G_s, we have

$$f(ars) = a^{\alpha} f(rs), \quad \forall r > 0,$$

where $\alpha > 0$, and $a = 2,3$. By letting $f_s(r) = f(rs)$, from Eaton (1966) and Gupta et al. (1988) on solutions of a Cauchy equation of one variable, we get $f_s(r) = f_s(1) r^{\alpha}$, $\forall r > 0$. It follows that for $x \in R^n$, $x \neq 0$,

$$f(x) = f\left(\frac{x}{\|x\|}\|x\|\right) = f_{\frac{x}{\|x\|}}(\|x\|) = f_{\frac{x}{\|x\|}}(1)\|x\|^{\alpha}.$$

If we let $g(s) = f_s(1)$ where s is an element of S, then g is a continuous function on S and

$$f(x) = g\left(\frac{x}{\|x\|}\right)\|x\|^{\alpha}, \quad \forall x \in R^n.$$

We obtained the general solution of the equation (1.1).

THEOREM 3.1. The general continuous solution f of the conditional Cauchy functional equation (1.1) is of the form

$$f(x) = g\left(\frac{x}{\|x\|}\right)\|x\|^{\alpha}, \quad \forall x \in R^n,$$

where g is an arbitrary continuous function on S.

An equivalent form of Theorem 3.1 is

THEOREM 3.1'. The general continuous solution f of the conditional Cauchy functional equation

$$f\left(a^{\frac{1}{\alpha}} x\right) = af(x), \quad \forall x \in R^n,$$

where $\alpha > 0$, and $a = 2,3$, is given by

$$f(x) = g\left(\frac{x}{\|x\|}\right)\|x\|^{\alpha}, \quad \forall x \in R^n,$$

where g is a continuous function on S.

4. A CHARACTERIZATION OF MULTIVARIATE STABLE DISTRIBUTIONS

Applying Theorem 3.1', we get the following characterization of multivariate stable distributions.

THEOREM 4.1. Let X_1, X_2, X_3 be independent and identically distributed, nondegenerate random vectors in R^n. Then X_1 has a multivariate stable distribution with $\mu = 0$ in (2.2) if and only if there is an α, $0 < \alpha \le 2$, such that $2^{1/\alpha}X_1$ and $X_1 + X_2$, $3^{1/\alpha}X_1$ and $X_1 + X_2 + X_3$ are identically distributed, respectively. In case $\alpha = 1$, all the univariate marginals of X_1 are Cauchy distributed.

PROOF. We need only show the sufficiency. Let ϕ be the characteristic function of X_1. We first show that $\phi(t) \neq 0$, $\forall t \in R^n$. Suppose there is $t_0 \in R^n$ such that $\phi(t_0) = 0$. Then the characteristic function of the univariate random variable $t_0'X_1$ is $\phi_{t_0'X_1}(u) = \phi(ut_0)$, with $\phi_{t_0'X_1}(1) = \phi(t_0) = 0$. In the proof of Theorem 3.1 of Gupta et al. (1988), it is shown that if $\phi_{t_0'X_1}(1) = 0$ then $\phi_{t_0'X_1}(u) = 0$ for all $-\infty < u < \infty$ and then $\phi(0) = 0$ which is a contradiction. Hence ϕ can be written as

$$\phi(t) = e^{h_1(t) + ih_2(t)},$$

or equivalently as,

$$\ln \phi(t) = h_1(t) + ih_2(t),$$

where h_1 and h_2 are real-valued continuous functions on R^n, the former is even and the latter is odd.

The hypotheses imply that for some α, $0 < \alpha \le 2$, $h_1(a^{1/\alpha}t) = ah_1(t)$ and $h_2(a^{1/\alpha}t) = ah_2(t)$, $\forall t \in R^n$, $a = 2,3$, and hence by Theorem 3.1', h_1 and h_2 are of the form

$$h_1(t) = g_1\left(\frac{t}{\|t\|}\right)\|t\|^{\alpha}, \; h_2(t) = g_2\left(\frac{t}{\|t\|}\right)\|t\|^{\alpha},$$

and

$$\ln \phi(t) = \|t\|^{\alpha}\left[g_1\left(\frac{t}{\|t\|}\right) + ig_2\left(\frac{t}{\|t\|}\right)\right].$$

Therefore by (2.2) X_1 has a multivariate stable distribution with location parameter vector $\mu = 0$.

In the case $\alpha = 1$ to show that all the univariate marginals of X_1 are Cauchy distributed, we are going to show that the first component of X_1 is Cauchy distributed, the Cauchy distributed of the other components of X_1 are obtained by a similar way.

Denoted by e_1 a vector of R^n having the first component equal to 1, and the other components equal to zero. The characteristic function ϕ_1 of the first component is obtained by giving $t = we_1$ in the characteristic function of X_1, for every $w \in R$, that is,

$$\ln \phi_1(w) = \ln \phi(we_1) = \|we_1\| \left[g_1\left(\frac{we_1}{\|we_1\|}\right) + ig_2\left(\frac{we_1}{\|we_1\|}\right)\right]$$
$$= |w| \left[g_1\left(\frac{w}{|w|} e_1\right) + ig_2\left(\frac{w}{|w|} e_1\right)\right].$$

By the fact that g_1 is an even function, g_2 is an odd function in the sense that $g_1(-t) = g_1(t)$, $g_2(-t) = -g_2(t)$, respectively, for every $t \in S$, then if $w > 0$, $\ln \phi_1(w) = w[g_1(e_1) + ig_2(e_1)] = wg_1(e_1) + iwg_2(e_1)$, and if $w < 0$, $\ln \phi_1(w) = -w[g_1(-e_1) + ig_2(-e_1)] = -w[g_1(e_1) - ig_2(e_1)] = -wg_1(e_1) + wg_2(e_1)$. Therefore for any $w \in R$, $\ln \phi_1(w) = |w| g_1(e_1) + iwg_2(e_1)$, and ϕ_1 is the characteristic function of a Cauchy distribution. As mentioned above, Press [7] showed that for any multivariate stable distribution, the characteristic function defined by (2.2) can be given by a simple explicit algebraic form, but Paulauska [6] pointed out that the form given by Press is only true for a class of multivariate stable distributions, and that the closed forms of g_1 and g_2 are still unknown, except for the case $\alpha = 2$, $g_2(t) = 0$, and $g_1(t) = t'\Sigma t$ for some positive definite $n \times n$ matrix Σ.

THEOREM 4.2. Let $X_1, X_2, \ldots, X_9$ be independent and identically distributed random vectors in R^n.

(i) X_1 has a multivariate normal distribution (possibly degenerate) with zero location vector if and only if $2X_1$ and $X_1 + X_2 + X_3 + X_4$, $3X_1$ and $X_1 + X_2 + \ldots + X_9$ are identically distributed, respectively.

(ii) X_1 has a multivariate stable distribution with Cauchy marginals if and only if $2X_1$ and $X_1 + X_2$, $3X_1$ and $X_1 + X_2 + X_3$ are identically distributed, respectively.

PROOF. The proof follows along the exact same lines as that of Theorem 4.1 by using Theorem 3.1 with $\alpha = 1$ or 2, instead of Theorem 3.1'.

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