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A NEW ORDERED COMPACTIFICATION

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ABSTRACT. A new Wallman-type ordered compactification $\gamma_o X$ is constructed using maximal CZ-filters (which have filter bases obtained from increasing and decreasing zero sets) as the underlying set. A necessary and sufficient condition is given for $\gamma_o X$ to coincide with the Nachbin compactification $\beta_o X$; in particular $\gamma_o X = \beta_o X$ whenever X has the discrete order. The Wallman ordered compactification $\omega_o X$ equals $\gamma_o X$ whenever X is a subspace of R^n. It is shown that $\gamma_o X$ is always T_1, but can fail to be T_1-ordered or T_2.

KEY WORDS AND PHRASES. CZ-set, maximal CZ-filter, T_1-ordered space, T_2-ordered space, Nachbin compactification, Wallman ordered compactification.

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0. INTRODUCTION.

L. Nachbin [10] initiated the study of ordered compactifications when he characterized the topological ordered spaces that allow T_2-ordered compactifications (we call these $T_{3.5}$-ordered spaces), and constructed the largest such T_2-ordered compactification $\beta_o X$ by embedding X in an ordered cube. The ***Nachbin*** (or ***Stone-Čech ordered***) ***compactification*** $\beta_o X$ has been studied and applied by various authors (see, for instance, our odd numbered references). A second ordered (but not necessarily T_2-ordered) compactification $\omega_o X$, called the ***Wallman ordered compactification***, was introduced by Choe and Park [2]. A necessary and sufficient condition for $\omega_o X = \beta_o X$ was given in [6], and in [8] the separation properties of $\omega_o X$ were investigated.

It is well known (e.g., see [4]) that the Stone-Čech compactification βX of a $T_{3.5}$ topological space can be described as a Wallman-type compactification using maximal filters of zero sets as the underlying set for the compactification. We have extended this construction to $T_{3.5}$-ordered spaces, and the result is a new ordered compactification which we call $\gamma_o X$. This new compactification, like $\beta_o X$ and $\omega_o X$, has the universal extension property for increasing, continuous maps into compact, T_2-ordered spaces. With the help of this universal property, we obtain necessary and sufficient conditions for $\gamma_o X = \beta_o X$; in particular this equality holds when the order of X is discrete. As an alternative approach to constructing $\beta_o X$, $\gamma_o X$ is more satisfactory than $\omega_o X$, in the sense that $\gamma_o X$ and $\beta_o X$ coincide on a larger class of spaces than do $\omega_o X$ and $\beta_o X$. Although we have not yet characterized the class of spaces for which $\gamma_o X = \omega_o X$, we have shown that this class includes all subspaces of R^n. This result enables us to show that $\gamma_o X$ can exhibit the same "pathological" behavior relative to separation properties that was demonstrated for $\omega_o X$ in [8].

For example, $\gamma_o X$ fails to be T_1-ordered if $X = R^n$ for $n \geq 3$.

It remains an open question whether $\beta_o X$ can be described via a Wallman-type ordered compactification for all $T_{3.5}$-ordered spaces X.

1. PRELIMINARIES.

Let $(X, \leq)$ be a poset and let A be a non-empty subset of X. Let $d(A) = \{x \in X : x \leq a \text{ for some } a \in A\}$ and $i(A) = \{x \in X : a \leq x \text{ for some } a \in A\}$, in case $A = \{x\}$, we write $d(x)$ and $i(x)$ rather than $d(\{x\})$ and $i(\{x\})$. The set A is said to be *decreasing* (respectively, *increasing*) if $A = d(A)$ (respectively, $A = i(A)$). A set which is either increasing or decreasing is said to be *monotone*; if $A = d(A) \cap i(A)$, then A is *convex*. If $f : (X, \leq) \to (Y, \leq)$ is a function between two posets, then f is *increasing* (respectively, *decreasing*) if $x \leq y$ in X implies $f(x) \leq f(y)$ (respectively, $f(y) \leq f(x)$) in Y.

A *topological ordered space* $(X, \leq, \tau)$ is a triple consisting of a poset $(X, \leq)$ and a convex topology τ on X; τ is *convex* if the open monotone sets form an open subbase. The term *space* will always mean topological ordered space, and $(X, \leq, \tau)$ will be shortened to X when there is no ambiguity. Note that every topological space can be regarded as a topological ordered space relative to the discrete order (equality).

Let E be the space $[0,1]$ with its usual order and topology. For an arbitrary space X we denote by $CI^*(X)$ (respectively, $CD^*(X)$) the set of all increasing (respectively, decreasing), continuous maps from X into E. An *increasing zero set* (respectively, *decreasing zero set*) is a set of the form $f^{-1}(0)$ where $f \in CD^*(X)$ (respectively, $f \in CI^*(X)$). The set of all increasing zero sets (respectively, decreasing zero sets) on X will be designated by $IZ(X)$ (respectively, $DZ(X)$). Using standard procedures described in [4], one easily proves the next two propositions.

PROPOSITION 1.1 If X is a space, $f \in CI^*(X)$, $g \in CD^*(X)$, and $a \in E$, then: (a) $f^{-1}([0,a]) \in DZ(X)$; (b) $f^{-1}([a,1]) \in IZ(X)$; (c) $g^{-1}([0,a]) \in IZ(X)$, and (d) $g^{-1}([a,1]) \in DZ(X)$.

PROPOSITION 1.2 For any space X, $IZ(X)$ and $DZ(X)$ are closed under countable intersections and finite unions.

A subset A of a space X is called a *C-zero set* (or CZ-set) if there is $B \in IZ(X)$ and $C \in DZ(X)$ such that $A = B \cap C$. Let $CZ(X)$ be the set of all CZ-sets on X. One easily verifies the following.

PROPOSITION 1.3 (a) $A \in CZ(X)$ iff there is $g \in CI^*(X)$ and $h \in CD^*(X)$ such that $A = f^{-1}(0)$, where $f = \frac{1}{2}(g+h)$. (b) The set $CZ(X)$ is closed under countable intersections.

It is generally not true that $CZ(X)$ is closed under finite unions; for instance, $[0,1]$ and $[2,3]$ are CZ-sets in R whose union is not a CZ-set.

By a *filter* $\mathcal{F}$ on X, we always mean a proper set filter (one that does not contain $\emptyset$). The filter on X generated by $\{x\}$, for $x \in X$, will be denoted by $\dot{x}$. If a filter $\mathcal{F}$ has a filter base of increasing zero sets, then $\mathcal{F}$ is called an *IZ-filter; DZ-filter* and *CZ-filter* are defined similarly. For an arbitrary filter $\mathcal{F}$ on X, let $IZ(\mathcal{F})$ (respectively, $DZ(\mathcal{F})$, $CZ(\mathcal{F})$) be the filter on X generated by $\mathcal{F} \cap IZ(X)$ (respectively, $\mathcal{F} \cap DZ(X)$, $\mathcal{F} \cap CZ(X)$). Note that $IZ(\mathcal{F})$ (respectively, $DZ(\mathcal{F})$, $CZ(\mathcal{F})$) is the finest IZ-filter (respectively, DZ-filter, CZ-filter) coarser than $\mathcal{F}$. The next proposition follows from Zorn's Lemma.

PROPOSITION 1.4 If $\mathcal{F}$ is a CZ-filter (respectively, IZ-filter, DZ-filter), there is a maximal CZ-filter (respectively, IZ-filter, DZ-filter) finer than $\mathcal{F}$.

PROPOSITION 1.5 Let X, Y be spaces and $f : X \to Y$ an increasing, continuous map.
(a) If $A \in IZ(Y)$ (respectively, $A \in DZ(Y)$, $A \in CZ(Y)$), then $f^{-1}(A) \in IZ(X)$ (respectively, $f^{-1}(A) \in DZ(X)$, $f^{-1}(A) \in CZ(X)$).
(b) If $\mathcal{F}$ is a filter on X, then $IZ(f(\mathcal{F})) \leq f(IZ(\mathcal{F}))$, $DZ(f(\mathcal{F})) \leq f(DZ(\mathcal{F}))$, and $CZ(f(\mathcal{F})) \leq f(CZ(\mathcal{F}))$.

PROOF. (a) If $A \in IZ(Y)$, then $A = g^{-1}(0)$, for $g \in CD^*(Y)$. Then $f^{-1}(A) = (g \circ f)^{-1}(0)$, where $g \circ f \in CD^*(X)$, and so $f^{-1}(A) \in IZ(X)$. The other cases are similar. (b) follows easily from (a). ∎

A space X is defined to be *T_1-ordered* if, for each $x \in X$, $i(x)$ and $d(x)$ are closed sets. A space X is *T_2-ordered* if, whenever $x \not\leq y$ in X, there is an increasing neighborhood U of x and a decreasing neighborhood V of y such that $U \cap V = \emptyset$; equivalently, $(X, \leq, \tau)$ is T_2-ordered if the order $\leq$ is a closed subset of $X \times X$. A space X is *$T_{3.5}$-ordered* if it satisfies the following conditions: (1) If $x \in X$, A is a closed subset of X, and $x \notin A$, then there is $f \in CI^*(X)$ and $g \in CD^*(X)$ such that $f(x) = g(x) = 0$ and $f(y) \vee g(y) = 1$ for $y \in A$; (2) If $x \not\leq y$ in X, there is an $f \in CI^*(X)$ such that $f(y) = 0$ and $f(x) = 1$. The $T_{3.5}$-ordered spaces are precisely the subspaces of compact, T_2-ordered spaces (see [10]). A space X is defined to be *T_4-ordered* if it is T_1-ordered and, whenever A and B are disjoint closed subsets with A decreasing and B increasing, there are disjoint open sets U and V, the former decreasing, the latter increasing, such that $A \subseteq U$ and $B \subseteq V$. Note that: compact and T_2-ordered $\Rightarrow$ T_4-ordered $\Rightarrow$ $T_{3.5}$-ordered $\Rightarrow$ T_2-ordered $\Rightarrow$ T_1-ordered. Also observe that T_1-ordered $\Rightarrow$ T_1, T_2-ordered $\Rightarrow$ T_2, and $T_{3.5}$-ordered $\Rightarrow$ $T_{3.5}$ (i.e., completely regular and T_1); it is not true, however, that T_4-ordered $\Rightarrow$ T_4.

In the remainder of this section we examine some properties of $T_{3.5}$-ordered spaces, with special emphasis on the role played by CZ-sets.

PROPOSITION 1.6 Let X be a $T_{3.5}$-ordered space. Let $x \in X$, and let $\mathcal{V}(x)$ be the filter of neighborhoods of x.

(a) $\mathcal{V}(x) = CZ(\mathcal{V}(x))$

(b) $\mathcal{V}(x)$ has an open base of sets of the form $(X - A) \cap (X - B)$, where $A \in CZ(X)$ and $B \in DZ(X)$.

(c) $CZ(X)$ is a closed subbase for X.

(d) If $\mathcal{F}$ is a filter on X such that $\mathcal{F} \to x$, then $CZ(\mathcal{F}) \to x$.

PROOF. (a) Let V be an open neighborhood of x. Then there are $f \in CI^*(X)$ and $g \in CD^*(X)$ such that $f(x) = g(x) = 0$ and $f(y) \vee g(y) = 1$ if $y \in X - V$. Then $f^{-1}([0, \frac{1}{2}]) \cap g^{-1}([0, \frac{1}{2}])$ is a CZ-set neighborhood of x which is a subset of V.

(b) Let f, g, and V be as in the proof of (a). If $B = f^{-1}(1)$ and $A = g^{-1}(1)$, then $A \in DZ(X)$, $B \in IZ(X)$, and $x \in (X - A) \cap (X - B) \subseteq V$.

(c) and (d) follow immediately from (b) and (a), respectively. ∎

PROPOSITION 1.7 In a $T_{3.5}$-ordered space X, the following statements are equivalent: (a) $x \leq y$; (b) $IZ(\dot{x}) \leq \dot{y}$; (c) $DZ(\dot{y}) \leq \dot{x}$.

PROOF. It is obvious that $(a) \Rightarrow (b)$. To show $(b) \Rightarrow (a)$, suppose y is in each member of $IZ(X)$ containing x, but $x \not\leq y$. Then there is $f \in CI^*(X)$ such that $f(y) = 0$ and $f(x) = 1$. Thus $y \notin f^{-1}(1)$, but $f^{-1}(1)$ is a member of $IZ(X)$ containing x. This establishes that $(a) \Leftrightarrow (b)$, and $(c) \Leftrightarrow (a)$ follows by a dual argument. ∎

In the next section we shall construct a compactification based on maximal CZ-filters. The next two propositions will be useful in this endeavor.

PROPOSITION 1.8 If X is a $T_{3.5}$-ordered space and $x \in X$, the $CZ(\dot{x})$ is the unique maximal CZ-filter on X coarser than $\dot{x}$.

PROOF. We already know that $CZ(\dot{x})$ is the finest CZ-filter coarser than $\dot{x}$. Suppose $\mathcal{G}$ is a CZ-filter and $CZ(\dot{x}) < \mathcal{G}$. Then there is a CZ-set $G \in \mathcal{G}$ such that $x \in X - G$. By Proposition 1.6(a), there is a CZ-neighborhood H of x such that $H \subseteq X - G$. Since $H \in CZ(\dot{x})$, the assumption that $CZ(\dot{x}) < \mathcal{G}$ is contradicted, and it follows that $CZ(\dot{x})$ is a maximal CZ-filter. It is obviously the only maximal CZ-filter coarser than $\dot{x}$. ∎

PROPOSITION 1.9 Let $f : X \to Y$ be a continuous, increasing map, where X is $T_{3.5}$-ordered and Y is compact and T_2-ordered. If $\mathcal{M}$ is a maximal CZ-filter on X, there is a unique point $y_{\mathcal{M}} \in Y$ such that $f(\mathcal{M}) \to y_{\mathcal{M}}$ in Y.

PROOF. Let $\mathcal{F}$ be an ultrafilter on X such that $\mathcal{M} \leq \mathcal{F}$. Since Y is compact and T_2, there is a unique point $y_{\mathcal{M}}$ in Y such that $f(\mathcal{F}) \to y_{\mathcal{M}}$. Because $\mathcal{M}$ is a maximal CZ-filter, $CZ(\mathcal{F}) \leq \mathcal{M}$, and $f(\mathcal{M}) \geq f(CZ(\mathcal{F})) \geq CZ(f(\mathcal{F}))$ follows by Proposition 1.5. But $f(\mathcal{F}) \to y_{\mathcal{M}}$ implies $CZ(f(\mathcal{F})) \to y_{\mathcal{M}}$ by Proposition 1.6(d), and therefore $f(\mathcal{M}) \to y_{\mathcal{M}}$. ∎

2. THE COMPACTIFICATION $\gamma_o X$

Throughout this section, we assume that X is a $T_{3.5}$-ordered space. An *ordered compactification* (Y, σ) of X is a pair consisting of a compact space Y and a map $\sigma : X \to Y$ such that σ is both a topological and an order embedding of X into Y such that $\sigma(X)$ is dense in Y. In this section, we shall construct an ordered compactification $(\gamma_o X, \psi)$ of X and establish some of its basic properties.

Let $\tilde{X}$ be the set of all maximal CZ-filters on X. By Proposition 1.8, these include all filters of the form $CZ(\dot{x})$, where $x \in X$. A relation $\precsim$ on $\tilde{X}$ is defined as follows: If $\mathcal{M}, \mathcal{N} \in \tilde{X}$, then $\mathcal{M} \precsim \mathcal{N}$ iff $IZ(\mathcal{M}) \leq \mathcal{N}$ and $DZ(\mathcal{N}) \leq \mathcal{M}$.

PROPOSITION 2.1 $(\tilde{X}, \precsim)$ is a poset.

PROOF. It is clear that $\precsim$ is reflexive and transitive. If $\mathcal{M} \precsim \mathcal{N}$ and $\mathcal{N} \precsim \mathcal{M}$, then $IZ(\mathcal{N}) \leq \mathcal{M}$ and $DZ(\mathcal{N}) \leq \mathcal{M}$. Since $\mathcal{N}$ is a CZ-filter, $IZ(\mathcal{N}) \vee DZ(\mathcal{N}) = \mathcal{N}$, and so $\mathcal{N} \leq \mathcal{M}$. It is also true that $IZ(\mathcal{M}) \leq \mathcal{N}$ and $IZ(\mathcal{N}) \leq \mathcal{M}$; therefore $\mathcal{M} \leq \mathcal{N}$, and we conclude that $\mathcal{M} = \mathcal{N}$. ∎

PROPOSITION 2.2 $x \leq y$ in X iff $CZ(\dot{x}) \precsim CZ(\dot{y})$ in $\tilde{X}$.

PROOF. If $x \leq y$, then by Proposition 1.7, $IZ(\dot{x}) = IZ(CZ(\dot{x})) \leq \dot{y}$, which implies $IZ(CZ(\dot{x})) \leq CZ(\dot{y})$. Likewise, $DZ(\dot{y}) \leq \dot{x}$, which implies $DZ(CZ(\dot{y})) \leq CZ(\dot{x})$. Thus $CZ(\dot{x}) \precsim CZ(\dot{y})$. This reasoning is reversible. ∎

For an arbitrary, non-empty subset A of X, we define $\tilde{A} = \{\mathcal{M} \in \tilde{X} : A \in \mathcal{M}\}$.

PROPOSITION 2.3 Let $A, B \in CZ(X)$.

(a) $\tilde{A} \cap \tilde{B} = \widetilde{A \cap B}$.

(b) $\tilde{A} \cup \tilde{B} = \widetilde{A \cup B}$.

(c) $\tilde{X} - \tilde{A} = \widetilde{X - A}$.

(d) If $A \in IZ(X)$, then $\tilde{A}$ is an increasing set in $\tilde{X}$.

(e) If $A \in DZ(X)$, then $\tilde{A}$ is a decreasing set in $\tilde{X}$.

PROOF. All of the assertions of this proposition are routine, and we shall verify only (d). If $\mathcal{M} \in \tilde{A}$ and $\mathcal{M} \precsim \mathcal{N}$, then $IZ(\mathcal{M}) \leq \mathcal{N}$. Since $A \in \mathcal{M}$ and $A \in IZ(X)$, $A \in IZ(\mathcal{M})$, and so $A \in \mathcal{N}$. Thus $\mathcal{N} \in \tilde{A}$, and $\tilde{A}$ is an increasing set. ∎

We next define $\psi : X \to \tilde{X}$ by $\psi(x) = CZ(\dot{x})$, for all $x \in X$. By Proposition 2.2, ψ is an order embedding of X in $\tilde{X}$. We omit the routine proof of the next proposition.

PROPOSITION 2.4 (a) For any $A \subseteq X$, $\psi^{-1}(\tilde{A}) \subseteq A$.
(b) If $A \in CZ(X)$, then $\psi^{-1}(\tilde{A}) = A$ and $\psi^{-1}(\widetilde{X - A}) = X - A$.

Let $\tilde{\tau}$ be the topology on $\tilde{X}$ with closed subbase $\{\tilde{A} : A \in CZ(X)\}$. From the two preceding propositions, it follows that $\tilde{\tau}$ has an open subbase of monotone open sets; thus $(\tilde{X}, \precsim, \tilde{\tau})$ is a topological ordered space. Let $\gamma_o X = (\tilde{X}, \precsim, \tilde{\tau})$.

THEOREM 2.5 For any $T_{3.5}$-ordered space X, $(\gamma_o X, \psi)$ is an ordered compactification for X whose topology is T_1.

PROOF. First note that $\psi : X \to \gamma_o X$ is a topological embedding by Propositions 1.6(c) and 2.4(b); ψ is also an order embedding, as we observed previously.

To show that $\gamma_o X$ is compact, it is sufficient to show that any collection $\mathcal{C} = \{\tilde{A}_i : A_i \in CZ(X), i \in I\}$ of subbasic closed sets in $\gamma_o X$ with the finite intersection property has a non-empty intersection. If $\mathcal{A} = \{A_i : i \in I\}$, then $\mathcal{A}$ has the finite intersection property by Proposition 2.3(a). Let $\mathcal{M}$ be any maximal CZ-filter containing $\mathcal{A}$; then $\mathcal{M} \in \bigcap \mathcal{C}$.

To show that γX_o is T_1, let $\mathcal{M}, \mathcal{N}$ be two distinct maximal CZ-filters on X. Then there are disjoint CZ-sets $M \in \mathcal{M}$ and $N \in \mathcal{N}$. It follows that $\widetilde{X - N}$ is a neighborhood of $\mathcal{M}$ not containing $\mathcal{N}$, and $\widetilde{X - M}$ is a neighborhood of $\mathcal{N}$ not containing $\mathcal{M}$.

Finally, if $\mathcal{M} \in \tilde{X}$, then $\psi(\mathcal{M})$ converges to $\mathcal{M}$ in $\gamma_o X$, and therefore $\psi(X)$ is dense in $\gamma_o X$. ∎

The next theorem shows that $\gamma_o X$ has the same universal extension property as $\omega_o X$ and $\beta_o X$.

THEOREM 2.6 Let X be a $T_{3.5}$-ordered space, Y a compact, T_2-ordered space, and $f : X \to Y$ be a continuous, increasing map. Then there is a unique continuous, increasing map $\tilde{f} : \gamma_o X \to Y$ such that the diagram below commutes.

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[dr, "f"'] & \gamma_o X \arrow[d, "\tilde{f}"] \\ & Y \end{tikzcd}$$

PROOF. Let $\tilde{f} : \gamma_o X \to Y$ be defined by $\tilde{f}(\mathcal{M}) = y_{\mathcal{M}}$, where $y_{\mathcal{M}}$ is defined in Proposition 1.9. We first show that $\tilde{f}$ is increasing. Let $\mathcal{M} \precsim \mathcal{N}$ in $\tilde{X}$; then $DZ(\mathcal{N}) \leq \mathcal{M}$.

Suppose $y_{\mathcal{M}} \not\leq y_{\mathcal{N}}$ in Y. Then there is $g \in CI^*(Y)$ such that $g(y_{\mathcal{M}}) = 1$ and $g(y_{\mathcal{N}}) = 0$. Thus $y_{\mathcal{N}} \in g^{-1}([0, \frac{1}{3}]) \in DZ(f(\mathcal{N}))$, since $f(\mathcal{N}) \to y_{\mathcal{N}}$ in Y. But $g^{-1}([\frac{2}{3}, 1]) \in f(\mathcal{M})$, since $f(\mathcal{M}) \to y_{\mathcal{M}}$, and therefore $f(\mathcal{M}) \not\geq DZ(f(\mathcal{N}))$. However, $DZ(\mathcal{N}) \leq \mathcal{M}$ implies $DZ(f(\mathcal{N})) \leq f(DZ(\mathcal{N})) \leq f(\mathcal{M})$ follows by Proposition 1.5. This contradiction establishes that $y_{\mathcal{M}} \leq y_{\mathcal{N}}$, and so $\tilde{f}$ is increasing.

We next show that $\tilde{f}$ is continuous. Let $\mathcal{M} \in \gamma_o X$ and let A be a CZ-neighborhood of $y_{\mathcal{M}}$ in Y. From the fact that $f(\mathcal{M}) \to y_{\mathcal{M}}$, we deduce that $\mathcal{M} \in \widetilde{f^{-1}(A)}$, and it is easy to see that $\tilde{f}(\widetilde{f^{-1}(A)}) \subseteq A$. It remains to show that $\widetilde{f^{-1}(A)}$ is a neighborhood of $\mathcal{M}$ in $\gamma_o X$. For this purpose, we employ Proposition 1.6(b) to obtain $C \in DZ(Y)$ and $D \in IZ(Y)$ such that $y_{\mathcal{M}} \in (Y - C) \cap (Y - D) \subseteq A$. Since $(Y - C) \cap (Y - D) \in f(\mathcal{M})$, it follows that $\mathcal{M} \in (\tilde{X} - \widetilde{f^{-1}(C)}) \cap (\tilde{X} - \widetilde{f^{-1}(D)})$. The latter set is open in $\gamma_o X$ and a subset of $\widetilde{f^{-1}(A)}$. This establishes that $\widetilde{f^{-1}(A)}$ is a neighborhood of $\mathcal{M}$ which maps into A, and the proof is complete. ∎

THEOREM 2.7 Let X be $T_{3.5}$-ordered. Then $\gamma_o X = \beta_o X$ iff the following conditions hold:

(1) If $M \in IZ(X)$, $N \in CZ(X)$, and $M \cap N = \emptyset$, then there is $h \in CI^*(X)$ such that $h(N) = 0$ and $h(M) = 1$.

(2) If $M \in DZ(X)$, $N \in CZ(X)$, and $M \cap N = \emptyset$, then there is $h \in CD^*(X)$ such that $h(M) = 0$ and $h(N) = 1$.

PROOF. Since $\beta_o X$ is the largest T_2-ordered compactification of X, Theorem 2.6 implies that $\gamma_o X = \beta_o X$ iff $\gamma_o X$ is T_2-ordered. Thus the proof will be achieved by showing that the specified conditions are necessary and sufficient in order for $\gamma_o X$ to be T_2-ordered.

Assume that $\gamma_o X$ is T_2-ordered and let M and N be as indicated in (1). By Proposition 2.3(a), $\tilde{M} \cap \tilde{N} = \emptyset$, and $\tilde{M}$ and $\tilde{N}$ are both closed subsets of $\gamma_o X$. Furthermore, $\tilde{M}$ is increasing in $\gamma_o X$ by Proposition 2.3(d). Let $d(\tilde{N})$ denote the decreasing hull of $\tilde{N}$ in $\gamma_o X$. Then $d(\tilde{N})$ is closed by Proposition 4, page 44, [10], and $d(\tilde{N}) \cap \tilde{M} = \emptyset$. By Theorem 1, page 30, [10], there is g in $CI^*(\gamma_o X)$ such that $g(\mathcal{M}) = 0$ if $\mathcal{M} \in d(\tilde{N})$ and $g(\mathcal{M}) = 1$ if $\mathcal{M} \in \tilde{M}$. Setting $h = g \circ \psi$, we obtain (1). A similar argument establishes (2).

Conversely, assume the two conditions, and let $\mathcal{M}, \mathcal{N}$ be elements of $\gamma_o X$ such that $\mathcal{M} \not\lesssim \mathcal{N}$. Then either $IZ(\mathcal{M}) \not\leq \mathcal{N}$ or $DZ(\mathcal{N}) \not\leq \mathcal{M}$. If $IZ(\mathcal{M}) \not\leq \mathcal{N}$, then (because $\mathcal{N}$ is a maximal CZ-filter) there is $M \in IZ(\mathcal{M})$ and a CZ-set $N \in \mathcal{N}$ such that $M \cap N = \emptyset$. If h is as stated in (1), then $h^{-1}\widetilde{([0, \frac{1}{2}))}$ and $h^{-1}\widetilde{((\frac{1}{2}, 1])}$ are disjoint open neighborhoods of $\mathcal{N}$ and $\mathcal{M}$ respectively, the former decreasing and the latter increasing. If $DZ(\mathcal{N}) \not\leq \mathcal{M}$, we can apply (2) to achieve the same result. ∎

If X has the discrete order, conditions (1) and (2) of Theorem 2.7 reduce to the statement that disjoint zero sets in X are "completely separated" in the sense of [4]. Since this is true for any $T_{3.5}$ space, we conclude that $\gamma_o X = \beta_o X = \beta X$ whenever X is a $T_{3.5}$-ordered space with the discrete order.

As we shall see in the next section, there are simple examples of $T_{3.5}$-ordered spaces for which $\gamma_o X$ is not T_2-ordered. In this case, we may be interested to know when $\gamma_o X$ satisfies the weaker separation properties "T_2" or "T_1-ordered". This section concludes with two theorems pertaining to this problem. Examples showing that $\gamma_o X$ need not satisfy these latter separation axioms are also provided in the next section.

THEOREM 2.8 Let X be a $T_{3.5}$-ordered space. Then $\gamma_o X$ is T_2 iff, for each ultrafilter $\mathcal{F}$ on X, there is a unique maximal CZ-filter $\mathcal{M}$ on X such that $CZ(\mathcal{F}) \leq \mathcal{M}$.

PROOF. Assume $\gamma_o X$ is T_2 and let $\mathcal{F}$ be an ultrafilter on X. Then $\psi(\mathcal{F})$ converges to some $\mathcal{M} \in \gamma_o X$, where $\mathcal{M}$ is a maximal CZ-filter on X. It must be true that $CZ(\mathcal{F}) \leq \mathcal{M}$; otherwise $\mathcal{M}$ and $CZ(\mathcal{F})$ would contain disjoint CZ-sets M and A, and $\widetilde{X - A}$ would be a neighborhood of $\mathcal{M}$ in $\gamma_o X$ not belonging to $\psi(\mathcal{F})$. If there were another maximal CZ-filter $\mathcal{N}$ finer than $CZ(\mathcal{F})$, then $\psi(\mathcal{F})$ would also converge to $\mathcal{N}$ in $\gamma_o X$, contradicting the assumption that $\gamma_o X$ is T_2. Thus $\mathcal{M}$ is the unique maximal CZ-filter such that $CZ(\mathcal{F}) \leq \mathcal{M}$.

Conversely, assume that $\gamma_o X$ is not T_2; then there is a filter $\mathcal{A}$ on $\gamma_o X$ converging to distinct elements $\mathcal{M}$ and $\mathcal{N}$ in $\gamma_o X$. Let $\mathcal{F}$ be an ultrafilter on X containing the filter base $\{A \subseteq X : \tilde{A} \in \mathcal{A}\}$. One easily verifies that $\psi(\mathcal{F})$ converges to both $\mathcal{M}$ and $\mathcal{N}$ in $\gamma_o X$. This implies, as in the preceding paragraph, that $\mathcal{M}$ and $\mathcal{N}$ are both maximal CZ-filters finer than $CZ(\mathcal{F})$, which contradicts the uniqueness condition. ∎

THEOREM 2.9 Let X be a $T_{3.5}$-ordered space such that, for each $A \in CZ(X)$, $i(A) \in IZ(X)$ and $d(A) \in DZ(X)$. Then $\gamma_o X$ is T_1-ordered.

PROOF. For $S \subseteq \gamma_o X$, let $i_\gamma(S)$ denote the increasing hull of S and $cl_\gamma S$ the closure of S in $\gamma_o X$. We will show that for arbitrary $\mathcal{M} \in \gamma_o X$, that $cl_\gamma(i_\gamma(\mathcal{M})) = i_\gamma(\mathcal{M})$, and hence $i_\gamma(\mathcal{M})$ is closed in $\gamma_o X$. The dual argument establishes that $d_\gamma \mathcal{M}$ is also closed.

First, observe that if $\mathcal{N} \in cl_\gamma(i_\gamma(\mathcal{M}))$, then for each $A \in CZ(X)$ such that $\mathcal{N} \in \widetilde{X - A}$, there is $\mathcal{K} \in i_\gamma(\mathcal{M})$ such that $\mathcal{K} \in \widetilde{X - A}$. In other words, if $\mathcal{N} \in cl_\gamma(i_\gamma(\mathcal{M}))$, then for each $A \in CZ(X)$ such that $A \notin \mathcal{N}$, there is $\mathcal{K} \in \gamma_o X$ such that $\mathcal{M} \lesssim \mathcal{K}$ and $A \notin \mathcal{K}$.

Let $\mathcal{N} \in cl_\gamma(i_\gamma(\mathcal{M}))$. If $\mathcal{N} \notin i_\gamma(\mathcal{M})$, then $\mathcal{M} \not\lesssim \mathcal{N}$, and so either $IZ(\mathcal{M}) \not\leq \mathcal{N}$ or $DZ(\mathcal{N}) \not\leq \mathcal{M}$. Assume the former; then there is $M \in \mathcal{M} \cap IZ(X)$ such that $M \notin \mathcal{N}$. But $\mathcal{N} \in cl_\gamma(i_\gamma(\mathcal{M}))$ implies there is $\mathcal{K} \in i_\gamma(\mathcal{M})$ such that $M \notin \mathcal{K}$. However $\mathcal{M} \lesssim \mathcal{K}$ implies $IZ(\mathcal{M}) \leq \mathcal{K}$, a contradiction. On the other hand, suppose $DZ(\mathcal{N}) \not\leq \mathcal{M}$. Since $\mathcal{M}$ is a maximal CZ-filter, there is a CZ-set $M \in \mathcal{M}$ and $N \in \mathcal{N} \cap DZ(X)$ such that $M \cap N = \emptyset$, and hence $N \cap i(M) = \emptyset$. But by assumption, $i(M) \in IZ(X)$, and so $i(M) \in IZ(\mathcal{M})$. Again, $\mathcal{N} \in cl_\gamma(i_\gamma(\mathcal{M}))$ implies there is $\mathcal{K} \gtrsim \mathcal{M}$ such that $i(M) \notin \mathcal{K}$. However $i(M) \in IZ(\mathcal{M}) \leq \mathcal{K}$ is again a contradiction. We therefore conclude that $i_\gamma(\mathcal{M})$ is closed in $\gamma_o X$. ∎

3. $\gamma_o X$ AND $\omega_o X$

The Wallman ordered compactification $(\omega_o X, \varphi)$ of a T_1-ordered space X was introduced by Choe and Park [2] in 1979. In this section we find conditions under which $\gamma_o X = \omega_o X$; this leads to examples showing that $\gamma_o X$ can fail, in various ways, to preserve the separation properties T_2, T_2-ordered, and T_1-ordered.

The construction of $\omega_o X$ and a discussion of its properties can be found in [8]. Here, we review only a few relevant facts. Although $\gamma_o X$ can be defined for any T_1-ordered space X, we shall assume, as in the preceding section, that X is $T_{3.5}$-ordered, since it is only for such spaces that $\gamma_o X$ and $\omega_o X$ can be compared.

If A is any non-empty subset of X, let $I(A)$ denote the smallest closed, increasing set containing A and $D(A)$ the smallest closed, decreasing overset of A. A is said to be a *c-set* if $A = I(A) \cap D(A)$. A space X is called a *c-space* if, for every c-set $A \subseteq X$, $i(A) = I(A)$ and $d(A) = D(A)$. A filter on X with a base of c-sets is called a c-filter. The underlying set for $\omega_o X$ is the set of all maximal c-filters on X. Indeed, the constructions of $\omega_o X$ and $\gamma_o X$ are very similar, with the c-sets playing the same role in the former that the CZ-sets play in the latter. In particular, if every c-set in X is a CZ-set, then $\omega_o X = \gamma_o X$. Thus the following proposition is obvious.

PROPOSITION 3.1 If every increasing closed set in X is in $IZ(X)$ and every decreasing closed set in X is in $DZ(X)$, then $\omega_o X = \gamma_o X$.

Another useful fact, proved in [6], is the following.

PROPOSITION 3.2 A space X has the property that $\omega_o X = \beta_o X$ iff X is a T_4-ordered c-space.

THEOREM 3.3 If X is a T_4-ordered space such that, for any sets F, G in $CZ(X)$, $I(F) \cap G = \emptyset$ implies $I(F) \cap D(G) = \emptyset$ and dually, then $\gamma_o X = \beta_o X$.

PROOF. We show that, under the given assumptions, X satisfies conditions (1) and (2) of Theorem 2.7. To verify (1), let $M \in IZ(X)$ and $N \in CZ(X)$ be disjoint. Since $I(M) = M$, it follows by our assumption that $M \cap D(G) = \emptyset$. Thus we can apply Nachbin's generalization of Urysohn's Lemma (see Theorem 1, page 30, [10]) to obtain $f \in CI^*(X)$ such that $f(M) = 1$ and $f(D(G)) = 0$. This establishes condition (1); the proof of (2) is similar. ∎

COROLLARY 3.4 If X is a $T_{3.5}$-ordered space such that $\omega_o X = \beta_o X$, then $\omega_o X = \gamma_o X$.

PROOF. If $\omega_o X = \beta_o X$, then, by Proposition 3.2, X is a T_4-ordered c-space. Every such space clearly satisfies the requirements of Theorem 3.3, and so the conclusion follows. ∎

A T_2-ordered space whose underlying partial order is a total (or linear) order is called a *totally ordered space*. It is shown in [7] that $\omega_o X = \beta_o X$ for any totally ordered space X.

COROLLARY 3.5 If X is a totally ordered space, then $\omega_o X = \gamma_o X = \beta_o X$.

THEOREM 3.6 Let X be a subspace of R^n. Then $\gamma_o X = \omega_o X$.

PROOF. In view of Proposition 3.1, it is sufficient to show that each closed, decreasing subset of X is in $DZ(X)$ and each closed, increasing subset of X is in $IZ(X)$.

We begin by defining (in the terminology of [3]) a quasi-pseudo-metric ρ on X defined as follows: If $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$, then $\rho(x, y) = (y_1 - x_1) \vee 0 + \cdots + (y_n - x_n) \vee 0$. If A is a non-empty, closed, decreasing subset of X, we define $\rho_A : X \to [0, \infty)$ as follows: $\rho_A(x) = \inf\{\rho(y, x) : y \in A\}$. Finally, let $h_A : X \to E$ be defined by $h_A = \rho_A \wedge 1$. It follows that $h_A \in CI^*(X)$ and $h_A^{-1}(0) = A$. Thus $A \in DZ(X)$. The dual argument shows that any closed, increasing subset of X is in $IZ(X)$.

It is shown in Theorem 3.4 of [8] that $\omega_o R^n = \beta_o R^n$ iff $n \leq 2$; this yields the following consequence of Theorem 3.6.

COROLLARY 3.7 $\gamma_o R^n = \beta_o R^n$ iff $n \leq 2$.

We recall two examples from [8] involving subspaces of R^2 in which $\omega_o X$, and hence also $\gamma_o X$, fail to exhibit basic separation properties. Let $S = \{(x,y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$ be a subspace of R^2. In Example 3.6 of [8], the subspace $X_1 = S - \{(0,0)\}$ of R^2 has the property that $\gamma_o X_1$ is neither T_1-ordered nor T_2. In Example 3.7 of [8], the subspace $X_2 = S - \{(0,y) : -1 \leq y \leq 1$ and $y \neq 0\}$ has the property that $\gamma_o X_2$ is T_2 but not T_1-ordered. We do not know of a space X for which $\gamma_o X$ is T_1-ordered but not T_2.

As a final example, recall that if X is a $T_{3.5}$-ordered space with the discrete order, then $\gamma_o X = \beta_o X$. If, in addition, X is chosen not to be T_4, then $\omega_o X$ (which in this case is the ordinary Wallman compactification) fails to be T_2, and consequently $\omega_o X \neq \gamma_o X$.

4. UNSOLVED PROBLEMS.

(1) Find necessary and sufficient conditions on a space X for $\gamma_o X$ to be T_1-ordered.

(2) Find conditions on a space X which are necessary and sufficient for $\gamma_o X = \omega_o X$.

(3) Determine whether $\gamma_o R^3$ is T_2.

(4) Find a $T_{3.5}$-ordered space X for which $\omega_o X$, $\beta_o X$, and $\gamma_o X$ are mutually non-equivalent.

(5) Determine whether $\beta_o X$ can be represented as a Wallman-type ordered compactification.

REFERENCES

[1] J. Blatter, "Order Compactification of Totally Ordered Spaces," *J. Approx. Theory* 13 (1975) 56-65.

[2] T. H. Choe and S. Park, "Wallman's Type Order Compactification," *Pacific J. Math.* 82 (1979) 339-347.

[3] P. Fletcher and W. Lindgren, *Quasi-Uniform Spaces*, Lecture Notes in Pure and Applied Mathematics, Vol. 77, Marcel Dekker, Inc., New York (1982).

[4] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton (1960).

[5] G. Hommel, "Increasing Radon Measures on Locally Compact Ordered Spaces," *Rendiconti Mathematica* 9 (1976) 85-117.

[6] D. C. Kent, "On the Wallman Order Compactification," *Pacific J. Math.* 118 (1985) 159-163.

[7] D. Kent and T. Richmond, "Ordered Compactification of Totally Ordered Spaces," *Internat. J. Math. & Math. Sci.* 11 (1988) 683-694.

[8] ____________ , "Separation Properties of the Wallman Ordered Compactification," *Internat. J. Math. & Math. Sci.* 13 (1990) 209-222.

[9] T. McCallion, "Compactifications of Ordered Topological Spaces," *Proc. Camb. Phil. Soc.* 71 (1972) 463-473.

[10] L. Nachbin, *Topology and Order*, Van Nostrand, New York Math. Studies, 4, Princeton, N.J. (1965).