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# DIVERGENT SEQUENCES SATISFYING THE LINEAR FRACTIONAL TRANSFORMATONS

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ABSTRACT. A real sequence $\{x_n\}_1^\infty$ which satisfies the recurrence $x_{n+1} = \frac{ax_n + b}{cx_n + d}$, in which all of $a, b, c, d$ are real will, for certain values of these constants, be divergent. It is the purpose of this note to examine the limit

$$\underset{N\to\infty}{Lim} \frac{1}{N} \sum_{n=1}^{N} f(x_n) : f \in C(-\infty, \infty)$$

in these cases. Except for certain exceptional values of $a, b, c, d$ this value is found for almost all $x_1$.

KEY WORDS AND PHRASES. Divergent series, linear fractional transformations, ergodic theory, Cesaro means.

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Consider the linear fractional transformation:

$$T(z) = \frac{az + b}{cz + d} \quad (ad - bc = 1 : c \neq 0)$$

which is to have the property of mapping the extended real axis $\mathbf{R}$ one-one onto itself. It is easily seen that a necessary and sufficient condition for this is that all of $a, b, c, d$ be real. (Note that we must allow $\mathbf{R}$ to contain the ideal "point at infinity" which we take as $\pm\infty$. This will allow the point $-\frac{d}{c}$ to have an image. Thus $\mathbf{R} \equiv [-\infty, +\infty)$.)

This transformation will have one or two fixed points. In the former case if $\lambda$ is the fixed point, it will be real and any sequence of real numbers satisfying $x_{n+1} = T(x_n)$ $(n = 1, 2, \cdots)$ will converge to $\lambda$. If there are two fixed points then either:

(a) they will both be real

or

(b) they will form a complex conjugate pair.

In case (a), any sequence generated as above will converge to one of the fixed points (call this one $\lambda$) whilst in case (b) there will not be convergence at all.

The behavior of the two convergent cases above can be expressed (more weakly) by asserting that

$$\frac{1}{N}\sum_{1}^{N} f(x_n) \to f(\lambda) \text{ as } N \to \infty$$

for any $f \in C(-\infty,\infty)$.

Our purpose here is to enquire about the existence and value of this limit in the non-convergent case (b) above. We shall show that, except for certain exceptional values of $a, b, c, d$, and for almost all $x_1$, we shall have

$$\frac{1}{N}\sum_{1}^{N} f(x_n) \to \int_{-\infty}^{+\infty} \frac{Kf(x)}{cx^2 + (d-a)x - b}\,dx \tag{1}$$

whenever $f$ is such that the right hand side exists as a Lebesgue integral. (The constant $K$ is a normalizing constant whose value is such as to make the right hand side unity when $f(x) \equiv 1$.)

If $\alpha$ and $\bar{\alpha}$ are the two fixed points in case (b) then the transformation

$$y = T(x) \equiv \frac{ax+b}{cx+d}$$

can be written as

$$\frac{y-\alpha}{y-\bar{\alpha}} = e^{i\beta}\,\frac{x-\alpha}{x-\bar{\alpha}} \tag{2}$$

In this $0 < \beta < 2\pi$ since, $c$ being non-zero, the identity transformation is excluded.
Define $S:\mathbf{R} \to [0,2\pi)$ as follows:

$S(x) = arg\left(\frac{x-\alpha}{x-\bar{\alpha}}\right)$, with the principal value of $arg$ lying in $[0,2\pi)$.

Define $H_\beta:[0,2\pi) \to [0,2\pi)$ by $H_\beta(\phi) = \beta + \phi \ (mod\ 2\pi)$
Clearly both $S$ and $H_\beta$ are one-one and onto. It is also well-known that, whenever $\beta$ is not a rational multiple of $\pi$, $H_\beta$ is ergodic with respect to ordinary Lebesgue probability measure $m$ on $[0,2\pi)$ and preserves this measure.

When $x$ and $y$ are real (2) is equivalent to

$$arg\left(\frac{y-\alpha}{y-\bar{\alpha}}\right) = \beta + arg\left(\frac{x-\alpha}{x-\bar{\alpha}}\right) \ (mod\ 2\pi)$$

That is,

$$S(y) = H_\beta S(x)$$

or

$$y = S^{-1} H_\beta S(x)$$

So, since $y = T(x) \equiv \frac{ax+b}{cx+d}$, we see now that $T \equiv S^{-1}H_\beta S$.

To deal with this situation we shall next prove a lemma (stated in somewhat more general terms than is required for the present application).

LEMMA. Let $M_1(X_1, \mathcal{A}, m_1)$ and $M_2(X_2, \mathcal{B}, m_2)$ be two measure spaces with probability measures $m_1$ and $m_2$. Let $S: X_1 \to X_2$ and $H: X_2 \to X_2$,, both of these transformations being one-one and onto. Suppose that $E \subset X_1$ is $m_1$-measurable if and only if $S(E) \subset X_2$ is $m_2$-measurable and that $m_1(E) = m_2(S(E))$. Finally, suppose that $H$ is $m_2$-measure preserving and ergodic with respect to this measure. Then the transformation $T = S^{-1}HS$ is $m_1$-measure preserving and ergodic with respect to $m_1$.

PROOF. Since all our transformations are invertible, we may deal with the transformations themselves rather than their inverses.

(a) Let $E \subset X_1$ be $m_1$-measurable.
Then $m_1(E) = m_2(S(E)) = m_2(HS(E)) = m_1(S^{-1}HS(E)) = m_1(T(E))$
This shows that $T$ is $m_1$-measure preserving.

(b) Let $A$ be a subset of $X_1$ which is invariant under $T$ and whose $m_1$ measure satisfies $0 < m_1(A) < 1$. Let $B = S(A)$. Then $B$ is an invariant set under $H$ because $H(B) = HS(A) = ST(A) = S(A) = B$. Also $m_1(A) = m_2(S(A)) = m_2(B)$ so that $0 < m_2(B) < 1$. But, since $H$ is ergodic, this is impossible. Hence there cannot exist any set $A$ satisfying both $T(A) = A$ and $0 < m_1(A) < 1$. This shows that $T$ is ergodic with respect to the $m_1$ measure and the proof of the lemma is complete.

We now return to the particular application in hand. As we have already remarked, when $\beta$ is not a rational multiple of $\pi$ the transformation $H_\beta$ of $[0,2\pi)$ onto itself is ergodic with respect to Lebesgue probability measure $m$ and preserves this measure. To apply the Lemma we let $X_1 \equiv \mathbf{R}, X_2 \equiv [0,2\pi), m_2 \equiv m$ (Lebesgue probability measure), $H \equiv H_\beta$ and $S \equiv$ the $S$ of the application. It remains to find $m_1$ so that $m_1(E) = m(S(E))$ whenever $E \subset \mathbf{R}$.

Write $\phi \equiv S(x) = arg\left(\frac{x-\alpha}{x-\bar{\alpha}}\right)$ so that $e^{i\phi} = \frac{x-\alpha}{x-\bar{\alpha}}$
By differentiation we find that

$$d\phi = \frac{\alpha-\bar{\alpha}}{i} \frac{1}{(x-\alpha)(x-\bar{\alpha})} dx$$

so that if $E \subset \mathbf{R} \equiv [-\infty,\infty)$ then

$$m(S(E)) = \frac{1}{2\pi} \int_{S(E)\cap[0,2\pi)} d\phi = \frac{\alpha-\bar{\alpha}}{2\pi i} \int_{E\cap[-\infty,\infty)} \frac{1}{(x-\alpha)(x-\bar{\alpha})} dx$$

Since $\alpha$ and $\bar{\alpha}$ are the zeros of $cx^2 + (d-a)x - b$ this latter integral can be re-written in an obvious way and so we find that the required $m_1$-measure is given by

$$m_1(E) = \int_{E\cap[-\infty,\infty)} \frac{K}{cx^2 + (d-a)x - b} dx$$

with $K$ chosen to make this a probability measure.

Applying the Lemma we see that the transformation $T$ preserves this measure and is ergodic with respect to it.

Finally, knowing this, we can apply the Individual Ergodic Theorem [1] to conclude the following:

THEOREM. Let $T(z) = \frac{az+b}{cz+d}$ $(ad-bc = 1 : c \neq 0)$ map the extended real axis $\mathbf{R}$ onto itself and let it have a pair of complex conjugate fixed points. If $a,b,c,d$ are such as to make $\beta$ in (2) not a rational multiple of $\pi$ then, for almost all $x_1$ (in the $m_1$ sense *), the sequence defined by

$$x_{n+1} = \frac{ax_n + b}{cx_n + d} \quad (n = 1,2,\cdots) \quad (ad-bc = 1, c \neq 0)$$

will have the property stated in (1) above.
NOTE: *By referring to the definition of $m_1$ measure above, "almost all in the $m_1$ sense" can be seen to be equivalent to "almost all in the sense of Lebesgue measure on $\mathbf{R}$."

## REFERENCES

1. PARRY, W., Topics in Ergodic Theory, Cambridge Univ. Press, 1981.