Internat. J. Math. & Math. Sci.
VOL. 17 NO. 1 (1994) 197-200

CR-HYPERSURFACES OF THE SIX-DIMENSIONAL SPHERE

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(Received July 23, 1992)

ABSTRACT. We proved that there does not exist a proper CR-hypersurface of S^6 with parallel second fundamental form. As a result of this we showed that S^6 does not admit a proper CR-totally umbilical hypersurface. We also proved that an Einstein proper CR-hypersurface of S^6 is an extrinsic sphere.

KEY WORDS AND PHRASES. Nearly Kaehler manifold, CR-submanifold, six-dimensional sphere, Einstein hypersurface totally umbilical.
1991 AMS SUBJECT CLASSIFICATION CODES. 53C40, 53C55.

1. INTRODUCTION.

It is known that of all the Euclidean spheres S^2 and S^6 admit the almost complex structure of which S^2 is complex and S^6 is not. It is also known that S^6 is an almost hermitian manifold which is nearly Kaehler but not Kaehler [4], that is, the almost complex structure is not parallel with respect to the Riemannian connection on S^6. Among all submanifolds of an almost Hermitian manifold, there are three typical classes; one is the class of holomorphic submanifold, one is the class of totally real submanifolds and the third is the class of CR-submanifolds. This last class was introduced by Bejancu [1]. Let $(\bar{M},J,g)$ be an almost Hermitian manifold with almost Hermitian structure (J,g) and M be a Riemannian submanifold of $\bar{M}$. The M is called a CR-submanifold of $\bar{M}$ if there exists a CR-holomorphic distribution D, i.e., $JD = D$ on M such that its orthogonal complement $D^{\perp}$ is totally real, i.e., $JD^{\perp} \subset \nu$ where ν is the normal bundle over M in $\bar{M}$. A CR-submanifold is called proper if neither $D = 0$, nor $D^{\perp} = 0$. The three classes of submanifolds of S^6, including CR-submanifolds, have been studied by several authors [2], [3], [5]. In this paper, we consider CR-hypersurfaces of S^6. We obtain the following results:

THEOREM 1. There does not exist a proper CR-hypersurface of S^6 with parallel second fundamental form.

THEOREM 2. S^6 does not admit a proper CR-totally umbilical hypersurface.

THEOREM 3. Let M be an Einstein proper CR-hypersurface of S^6, then M is an extrinsic sphere.

PRELIMINARIES. Let $(\bar{M},g)$ be a Riemannian manifold and M be a Riemannian submanifold of $\bar{M}$. Let ∇ (resp. $\bar{\nabla}$) be the Riemannian connection on M (resp. $\bar{M}$) and R (resp. $\bar{R}$) be the curvature tensor of M (resp. $\bar{M}$). Denote by h the second fundamental form of M in $\bar{M}$. Then the Gauss formula and the Weingarten formula are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{1.1}$$

$$\bar{\nabla}_X N = -A_N X + \overset{\perp}{\nabla}_X N \qquad \begin{array}{c} X,Y \in \mathfrak{X}(M) \\ N \in \nu \end{array} \tag{1.2}$$

where $-A_N X$ (resp. $\overset{\perp}{\nabla}_X N$) denotes the tantial part (resp. normal part) of $\bar{\nabla}_X N$. The tangential component $A_N X$ is related to the second fundamental form by

$$g(h(X,Y),N) = g(A_N X, Y), \qquad X,Y \in \mathfrak{X}(M).$$

The Gauss equation is given by

$$g(R(X,Y)Z,W) = g(\bar{R}(X,Y)Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) \tag{1.3}$$

The Codazzi equation is

$$g(\bar{R}(X,Y)Z,N) = g((\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z),N) \tag{1.4}$$

where

$$(\bar{\nabla}_X h)(Y,Z) = \overset{\perp}{\nabla}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

If $(e_i) i = 1,2,\cdots,\eta$ is a frame field for M, then the Ricci curvature S of M is given by

$$S(X,Y) = \sum_{i=1}^{n} R(e_i, X, Y, e_i).$$

The submanifold M is called an Einstein manifold if $S(X,Y) = cg(X,Y)$ for some constant c and any $X,Y \in \mathfrak{X}(M)$. M is said to be totally umbilical if $h(X,Y) = g(X,Y)H$ where H is the mean curvature vector defined by $H = \frac{1}{n}$ trace h.

M is called an extrinsic sphere if $\overset{\perp}{\nabla}_X H = 0$ for any $X \in \mathfrak{X}(M)$. The CR-submanifold M is called a CR-product submanifold if it is locally the Riemannian product of a holomorphic submanifold and a totally real submanifold. Sekigawa [6] proved that in S^6 there does not exist any CR-product submanifolds.

2. PROOF OF THE MAIN RESULTS.

PROOF OF THEOREM 1. Since the second fundamental form is parallel we have $(\nabla_W A)(V) = 0$ or $\nabla_W AV = A\nabla_W V$ for any V, $W \in \mathfrak{X}(M)$. If V is an eigenvector of A with corresponding eigenvalue β, i.e., $AV = \beta V$, then from the equation $\nabla_W(AV) = A\nabla_W V$ we get $\beta\nabla_W V = A\nabla_W V$. This means that $\nabla_W V$ is an eigenvector corresponding to eigenvalue β whenever V is. If T is the eigenspace of β then $\nabla_W T \subset T$.

Since M is a proper CR-hypersurface of S^6, we can take $\{E_1, JE_1, E_2, JE_2, \xi\}$ as an orthonormal frame field for TM where $E_1, E_2 \in D$ and $\xi \in D^{\perp}$. Also since the normal bundle is 1-dimensional we assume that the frame $\{E_1, JE_1, E_2, JE_2, \xi\}$ diagonalizes A. So let $AE_1 = \alpha_1 E_1$, $AJE_1 = \bar{\alpha}_1 JE_1$, $AE_2 = \alpha_2 E_2$, $AJE_2 = \bar{\alpha}_2 JE_2$ and $A\xi = \beta\xi$. We consider the two cases for the eigenvalues $\alpha_i, \bar{\alpha}_i, \beta$ $i = 1,2$.

CASE 1: $\alpha_i \neq \beta$ and $\bar{\alpha}_i \neq \beta$ for all $i = 1,2$.

In this case we have $g(\nabla_W \xi, E_i) = g(\nabla_W \xi, JE_i) = 0$ for all $W \in \mathfrak{X}(M)$. This gives $\nabla_W \xi \in D^{\perp}$, i.e., the distribution $D^{\perp}$ is parallel. Since $\nabla_W \xi \in D$ we get $\nabla_W \xi = 0$. This last equation with $g(\xi, X) = 0$ for $X \in D$ gives $\nabla_W X \in D$, i.e., the distribution D is also parallel. This implies that M is a CR-product, a contradiction, since S^6 does not admit any CR-product submanifold [6].

CASE 2: $\alpha_{i_o} = \beta$ or $\bar{\alpha}_{i_o} = \beta$ for some i_o.

Without loss of generality let us assume that $\alpha_{i_o} = \beta$ for some i_o. Then the space T spanned by $\{E_{i_o}, \xi\}$ is the eigenspace of eigenvalue $\beta = \alpha_{i_o}$. We then have $\nabla_W T \subset T$. In particular

$\nabla_{E_{i_o}} \xi = aE_i + b\xi$ for some functions a and b. Since $g(\nabla_{E_{I_o}} \xi, \xi) = 0$, we get $\nabla_{E_{i_o}} \xi = aE_{i_o}$. Also using the equation $\bar{\nabla}_{E_{i_o}} JE_{i_o} = J\bar{\nabla}_{E_{i_o}} E_{i_o}$ with the help of equations (1.1) and (1.2) and the fact that $h \in JD^{\perp}$ we get $g(\nabla_{E_{i_o}} E_{i_o}, \xi) = 0$. From which we get $g(\nabla_{E_{i_o}} \xi, E_{i_o}) = 0$, i.e., $\nabla_{E_{i_o}} \xi = 0$. Now using this last equation and the fact that $\nabla_{\xi} \xi = 0$, we get

$$R(E_{i_o}, \xi)\xi = \nabla_{E_{i_o}} \nabla_{\xi} \xi - \nabla_{\xi} \nabla_{E_{i_o}} \xi - \nabla_{[E_{i_o}, \xi]} \xi = \nabla_{\nabla_{\xi} E_{i_o}} \xi .$$

But $\nabla_{\xi} E_{i_o} = cE_{i_o} + d\xi = 0$ since $g(\nabla_{\xi} E_{i_o}, E_{i_o}) = 0$. $g(\nabla_{\xi} E_{i_o}, \xi) = -g(\nabla_{\xi} \xi, E_{i_o}) = 0$. So $R(E_{i_o}, \xi)\xi = 0$. However from Gauss equation we obtain $g(R(E_{i_o}, \xi)\xi, E_{i_o}) = c + \beta^2 > 0$ which is a contradiction. This finishes the proof of Theorem 1.

PROOF OF THEOREM 2. Since M is totally umbilical we have $h(X,Y) = g(X,Y)H$ for any $X, Y \in \mathfrak{X}(M)$. Using this in Codazzi equation (1.4) we get $g(R(X,Y)Z, N) = g(g(Y,Z)\overset{\perp}{\nabla}_X H - g(X,Z)\overset{\perp}{\nabla}_Y H, N)$. Since the ambient space S^6 is of constant curvature we have $g(g(Y,Z)\overset{\perp}{\nabla}_X H - g(X,Z)\overset{\perp}{\nabla}_Y H, N) = 0, X, Y, Z \in \mathfrak{X}(M)$. Now for any $X \in \mathfrak{X}(M)$, choose Y such that $g(Y,X) = 0$ and let $Z = Y$. Then the above equation gives $\overset{\perp}{\nabla}_X H = 0$, i.e., H is parallel. Using a frame field $(E_i), 1 \le i \le 5$ with E_5 in $D^{\perp}$ and the rest in D, one can write $H = \gamma JE_5$ for some constant γ. Also the equation $h(X,Y) = g(X,Y)H$ gives $h(E_i, E_i) = \gamma JE_5$, and $h(E_i, E_j) = 0$ for $i \neq j$. Note that in this case

$$\begin{aligned}(\bar{\nabla}_{E_i} h)(E_j, E_k) &= \overset{\perp}{\nabla}_{E_i} h(E_j, E_k) - h(\nabla_{E_i} E_j, E_k) - h(E_j, \nabla_{E_i} E_k) \\ &= E_i g(E_j, E_k)H = 0 \text{ for all } i, j.\end{aligned}$$

where we have used the equation $h(X,Y) = g(X,Y)H$ in the second equality. This means that M has parallel second fundamental form. Then using Theorem 1 we obtain Theorem 2.

PROOF OF THEOREM 3. Let $\{X, Y, JX, JY, Z\}$ be an orthonormal frame for TM where $X, Y \in D$ and $Z \in D^{\perp}$. Since M is a hypersurface we know that the above frame diagonalizes A. Therefore one can write

$$h(Z,Z) = \alpha JZ, h(X,X) = \beta JZ, h(JX,JX) = \gamma JZ, h(Y,Y) = \delta JZ, h(JY,JY) = \eta JZ$$

and

$$h(Z,X) = h(Z,JX) = h(Z,Y) = h(Z,JY) = h(X,JX) = h(X,Y) = h(X,JY) = h(Y,JY) = 0$$

where $\alpha, \beta, \gamma, \delta, \eta$ are smooth functions on M. Then using Guass equation (1.3) we get

$$S(Z,Z) = R(X,Z,Z,X) + R(JX,Z,Z,JX) + R(Y,Z,Z,Y) + R(JY,Z,Z,JX) = 4c + \alpha(\beta + \gamma + \delta + \eta)$$

Similarly

$$S(X,X) = 4c + \beta(\alpha + \gamma + \delta + \eta)$$

$$S(JX,JX) = 4c + \gamma(\alpha + \beta + \delta + \eta)$$

$$S(Y,Y) = 4c + \delta(\alpha + \beta + \gamma + \eta)$$

$$S(JY,JY) = 4c + \eta(\alpha + \beta + \delta + \eta)$$

Since M is Einstein we have

$$S(Z,Z) = S(X,X) = S(JX,JX) = S(Y,Y) = S(JY,JY) = constant$$

i.e.,

$$\underset{\text{(i)}}{\alpha(\beta + \gamma + \delta + \eta)} = \underset{\text{(ii)}}{\beta(\alpha + \gamma + \delta + \eta)} = \underset{\text{(iii)}}{\gamma(\alpha + \beta + \delta + \eta)} = \underset{\text{(iv)}}{\delta(\alpha + \beta + \gamma + \eta)} = \underset{\text{(v)}}{\eta(\alpha + \beta + \gamma + \delta)} = const.$$

We shall show that $\alpha, \beta, \gamma, \delta$ and η are constants. From the above equations we have:

$$\alpha(\gamma+\delta+\eta)=\beta(\gamma+\delta+\eta), \qquad \beta(\alpha+\delta+\eta)=\gamma(\alpha+\delta+\eta)$$

$$\gamma(\alpha+\beta+\eta)=\delta(\alpha+\beta+\eta), \qquad \delta(\alpha+\beta+\gamma)=\eta(\alpha+\beta+\gamma)$$

We seek all solutions for this system. One obvious solution is $\alpha=\beta=\gamma=\delta=\eta=const.$ The other possible solutions are the following cases:

(a) $\gamma+\delta+\eta=\alpha+\delta+\eta=\alpha+\beta+\eta=\alpha+\beta+\gamma=0$
In this case we have $\alpha=\gamma=\eta=const.$ and $\delta=\beta$ considering (i) and (iv) with $\delta=\beta$ we get $\delta=\beta=\alpha=const.$ or $\delta=\beta=-2\alpha=const.$ So for this case $\alpha,\beta,\gamma,\delta,\eta$ are constants.

(b) $\alpha=\beta, \alpha+\delta+\eta=\alpha+\beta+\eta=\alpha+\beta+\gamma=0$, i.e., $\alpha=\beta=\delta$ and $\eta=\gamma$. Using (ii) and (v) with $\alpha=\beta=\delta$ we get $\eta=\gamma=\alpha=const.$ or $\eta=\gamma=-2\alpha=const.$, i.e., $\alpha,\beta,\gamma,\delta,\eta$ are constants.

(c) $\alpha=\beta=\gamma, \alpha+\beta+\eta=\alpha+\beta+\gamma=0$, i.e., $\alpha=\beta=\gamma=\eta$. Using (i) and (iv) with this last equation we get $\alpha=\beta=\gamma=\eta=\delta=const.$ (Note that in case $\alpha=\beta=\gamma=\delta=0$, then M is totally geodesic and hence $\delta=0$).

(d) $\alpha=\beta=\gamma=\delta, \alpha+\beta+\gamma=0$, i.e., $\alpha=\beta=\gamma=\delta=0$. Following the note in (c) we have $\eta=0$.

Therefore in all cases $\alpha=\beta=\gamma=\delta=\eta=const.$ We conclude that $H=\alpha JZ$ where α is constant and thus $\nabla_V^{\perp} H=0$ for any $V \in \mathfrak{X}(M)$, i.e., M is an extrinsic sphere.

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