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# RADICAL CLASSES OF *l*-GROUPS

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ABSTRACT. The main results of this paper concern radical classes of $l$-groups. In the sections 2-3 the relationship between several radical classes of $l$-groups are discussed and the characteristic properties for several radical mappings are given. In the sections 5-6 we give nice concrete descriptions of some important radical classes of $l$-groups using the structure theorems of a complete $l$-group and an Archimedean $l$-group.

KEY WORDS AND PHRASES. Lattice ordered groups ($l$-groups), radical classes of $l$-groups, product radical classes, subproduct radical classes, radical mappings.

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## 1. INTRODUCTION

An $l$-group G is a group that is also a lattice such that $c+a+d\leqslant c+b+d$ whenever $a\leqslant b$ [1]. The theory of $l$-groups is as natural as that of rings. But the fact of G is equipped with two different kind of operations makes the things more complecated. We have more subobjects in the category of $l$-groups. An $l$-subgroup of an $l$-group is both a subgroup and a sublattice. An $l$-subgroup $H$ is convex if $a$, $b\in H$ and $a<g<b$ imply that $g\in H$. A normal convex $l$-subgroup is called an $l$-ideal. A function $\varphi$: $G\to H$ between $l$-groups $G$ and $H$ is an $l$-homomorphism if it is a group and a lattice homomorphism. Let $\{G_a \mid a\in A\}$ be a family of $l$-groups and $\prod\limits_{a\in A} G_a$ be their direct product where $(----g_a----) \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} (----f_a----) = (----g_a \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} f_a----)$. An $l$-group $G$ is said to be a subdirect product of $G_a$, in symbols $G\subseteq' \prod\limits_{a\in A} G_a$, if $G$ is an $l$-subgroup of $\prod\limits_{a\in A} G_a$ such that for each $a\in A$ and each $g'\in G_a$ there exists $g\in G$ with the property $g_a=g'$. We denote the $l$-subgroup of $\prod\limits_{a\in A} G_a$ consisting of the elements with only finitely many non-zero components by $\sum\limits_{a\in A} G_a$. It is called the direct sum of $\{G_a \mid a\in A\}$. An $l$-group $G$ is said to be a completely subdirect product of $G_a$, if $G$ is an $l$-subgroup of $\prod\limits_{a\in A} G_a$ and $\sum\limits_{a\in A} G_a\subseteq G$. An $l$-group $G$ is said to be an ideal subdirect product of $G_a$, in symbols $G\subseteq^* \prod\limits_{a\in A} G_a$, if $G\subseteq' \prod\limits_{a\in A} G_a$ and $G$ is an $l$-ideal of $\prod\limits_{a\in A} G_a$.

Let $G$ be an $l$-group and $X\subseteq G$. $X_G^{\perp}=\{f\in G \mid \text{for all } x\in X,\ |f|\wedge|x|=0\}$ is called the polar of $X$ in $G$ and $X^{\perp\perp}=(X^{\perp})^{\perp}$ is called the double polar. An $l$-subgroup $H$ of $G$ is closed in $G$ if, for all subsets $\{x_a \mid a\in A\}$ of $H$ such that $a=\bigvee\limits_{a\in A} x_a$ exists in $G$ we have $a\in H$. The order closure $\overline{H}_G$ of $H$ in $G$ is the smallest closed $l$-subgroup of $G$ containing $H$. Let $G_\lambda$ ($\lambda\in \Lambda$) be convex $l$-subgroups of $G$. The join $\bigvee\limits_{\lambda\in\Lambda} G_\lambda$ is the smallest convex $l$-subgruop of $G$ containing $G_\lambda$ ($\lambda\in\Lambda$).

A variety of any type of algebras is an equationally defined class. It is an important area in the study of algebras. In 1935 G. Birkhoff proved that a class of algebras is a variety exactly if it is closed under the formation of subalgebras, products and homomorphic images [2]. In 1937 B. H. Neumann initiated their study for varieties of groups [3, 4]. In the early 70's J. Martinez began the study of varieties of $l$-groups [5, 6]. He also studied torsion classes of $l$-groups [7, 8, 9]. J. Jakubik studied radical classes of $l$-groups [10, 11, 12, 13, 14]. In this paper we give some results in the study for radical classes of $l$-groups. We use the standard terminologies and notations of [1, 15, 16].

We can make new $l$-groups from some original $l$-groups. These structures include:

1. taking $l$-subgroups,

1′. taking convex $l$-subgroups,

2. forming joins of convex $l$-subgroups,

3. forming completely subdirect products,

3′. forming direct products,

3″. forming direct sums,

4. taking $l$-homomorphic images,

4′. taking complete $l$-homomorphic images,

4″. taking $l$-isomorphic images,

5. forming extensions, that is, $G$ is an extension of $A$ by using $B$ if $A$ is an $l$-ideal of $G$ and $B=G/A$,

6. taking order closures, that is, $G$ is an order closure of $A$ if $A$ is a convex $l$-subgroup of an $l$-group $H$ and $G=\overline{A}_H$.

7. taking double polars, that is, $G$ is a double polar of $A$ if $A$ is a convex $l$-subgroup of an $l$-group $H$ and $G=A_H^{\perp\perp}$.

A family $\mathscr{U}$ of $l$-groups is called a class, if it is closed under some structures. If a class $\mathscr{U}$ is closed under the structures $i_1, \cdots\cdots, i_k$, we call $\mathscr{U}$ $i_1\cdots\cdots i_k$-class where $i_1, \cdots\cdots, i_k \in \{1, 1', 2, 3, 3', 4, 4', 4'', 5, 6, 7\}$ and $1 \leqslant k \leqslant 7$. All our classes always assumed to contain along with a given $l$-group all its $l$-isomorphic images, so we omit the index 4″. Thus, a radical class [10] is a 1′2-class, a quasi-torsion class [17] is a 1′24′-class, a torsion class [7] is a 1′24-class, a s-closed radical class [18] is a 12-class, a closed-kernel radical class [18] is a 1′26-class, a polar kernel radical class [18] is a 1′27-class, a variety [19] is a 13′4-class. 1′25-class is called a complete (or idempotent) radical class. We call a 1′23′-class (1′23-class) a product radical class (a subproduct radical class). In this paper we call all $1'2i_3\cdots\cdots i_k$-classes radical classes where $i_3, \cdots\cdots, i_k \in \{3, 3', 3'', 4, 4', 5, 6, 7\}$.

## 2. THE RELATIONSHIP BETWEEN RADICAL CLASSES

Let $\mathscr{R}$ be a radical class and $G$ be an $l$-group. Then there exists a largest convex $l$-subgroup of $G$ belonging to $\mathscr{R}$. We denote it by $\mathscr{R}(G)$ and call $\mathscr{R}(G)$ the $\mathscr{R}$-radical of G. It is invariant under all the $l$-automorphisms of G. Let $T_{i_1\cdots\cdots i_k}$ be the set of all $i_1\cdots\cdots i_k$- classes.

LEMMA 2. 1. $T_{1'2}=T_{1'23''}$.

Proof. It suffices to prove that each radical class is closed under forming direct sums. Suppose that $\mathscr{U}$ is a radical class and $\{G_a \mid a\in A\} \subseteq \mathscr{U}$. Consider $G=\prod_{a\in A} G_a$. Let $\overline{G}_a=\{f \in \prod_{a\in A} G_a \mid a'\neq a \Rightarrow f_{a'}=0\}$ for each $a\in A$. Then $\sum_{a\in A} G_a \cong \bigvee_{a\in A}^{(G)} \overline{G}_a$. Since $\mathscr{U}$ is closed under forming joins of convex $l$-subgruops, $\sum_{\alpha\in A} G_\alpha \in \mathscr{U}$.

A radical class $\mathscr{R}$ is said to be a closed-kernel radical class if for any $l$-group $G$ $\mathscr{R}(G)$ is closed [18].

LEMMA 2. 2. A radical class $\mathscr{R}$ is closed-kernel if and only if $\mathscr{R}$ is colsed under taking order closures.

Proof. Suppose that $\mathscr{R}$ is a closed-kernel radical class, that is $\mathscr{R}(G) = \overline{\mathscr{R}(G)_G}$for any $l$-group $G$. Let $G\in\mathscr{R}$ and $\overline{G_H}$ is an order closure of $G$ in an $l$-group $H$, $G\subseteq\overline{G}_H$. Then $G\subseteq\mathscr{R}(\overline{G}_H)\subseteq\overline{G}_H$. So $\mathscr{R}(\overline{G_H}) = \overline{\mathscr{R}(\overline{G_H})} = \overline{G_H}$ and $\overline{G_H}\in\mathscr{R}$. Conversely, suppose that a radical class $\mathscr{R}$is closed under taking order closures. Then for any $l$-group $G$, $\mathscr{R}(G)\in\mathscr{R}$ implies $\overline{\mathscr{R}(G)_G}\in\mathscr{R}$. Since $\mathscr{R}(G)$ is the largest convex $l$-subgraoup of $G$ belonging to $\mathscr{R}$, $\mathscr{R}(G) = \overline{\mathscr{R}(G)_G}$.

LEMMA 2. 3. Every closed-kernal radical class is also a subproduct radical class, that is $T_{1'26} = T_{1'23}$.

Proof. Suppose that $\mathscr{R}$ is a closed-kernel radical class and $G$ is a completely subdirect product of $\{G_\lambda \mid \lambda\in\Lambda\}$ where $\{G_\lambda\mid\lambda\in\Lambda\}\subseteq\mathscr{R}$, that is

$$\sum_{\lambda\in\Lambda} G_\lambda \subseteq G \subseteq' \prod_{\lambda\in\Lambda} G_\lambda.$$

Then $\mathscr{R}(G)\cap\overline{G_\lambda} = \mathscr{R}(\overline{G_\lambda}) = \overline{G}_\lambda$ and so $G\supseteq\mathscr{R}(G)\supseteq\overline{G}_\lambda$ for each $\lambda\in\Lambda$. Let $a = (\text{----}, a_\lambda, \text{----})\in G$. Then

$$a = \bigvee_{\lambda\in\Lambda}{}^{(G)}\bar{a}_\lambda$$

where $\bar{a}_\lambda = (0, \cdots\cdots, 0, a_\lambda, 0, \cdots\cdots, 0)\in\overline{G}_\lambda\ (\lambda\in\Lambda)$。Since $\mathscr{R}$ is closed-kermel, $a\in\mathscr{R}(G)$. Hence $G=\mathscr{R}(G)$ and $G\in\mathscr{R}$.

A radical class $\mathscr{R}$ is called a polar kernel radical class if $\mathscr{R} = \mathscr{R}^{\perp\perp}$, that is $\mathscr{R}(G) = \mathscr{R}(G)^{\perp\perp}$ for any $l$-group $G$.

LEMMA 2. 4. A radical class $\mathscr{R}$ is a polar kernel radical class if and only if $\mathscr{R}$ is closed under taking double polars.

Proof. Suppose that $\mathscr{R}$ is a polar kernal radical class. Let $G\in\mathscr{R}$ and $G_H^{\perp\perp}$ is a double polar of $G$ in an $l$-group $H$. Then $G\subseteq\mathscr{R}(G_H^{\perp\perp})\subseteq G_H^{\perp\perp}$ and $G_H^{\perp\perp}\subseteq\mathscr{R}(G_H^{\perp\perp})\subseteq G_H^{\perp\perp}$. So $\mathscr{R}(G_H^{\perp\perp}) = G_H^{\perp\perp}$ and $G_H^{\perp\perp}\in\mathscr{R}$. Conversely, suppose that a radical class $\mathscr{R}$ is closed under taking double polars. Then for any $l$-group $G$, $\mathscr{R}(G)\in\mathscr{R}$ implies $\mathscr{R}(G)_G^{\perp\perp}\in\mathscr{R}$. But $\mathscr{R}(G)$ is the largest convex $l$-subgroup of $G$ belonging to $\mathscr{R}$, so $\mathscr{R}(G) = \mathscr{R}(G)_G^{\perp\perp}$.

If $\mathscr{R}$ and $\mathscr{T}$ are two $1'2$-classes, define the prodict $\mathscr{R}.\mathscr{T} = \{G\mid G/\mathscr{R}(G)\in\mathscr{T}\}$. $\mathscr{R}.\ \mathscr{T}$ is then a $1'2$-class. Now similarly to [7] we give a more description of complete $1'2$-classes. Let $\mathscr{T}$ be a $1'2$-class and $\sigma$ be an ordinal number. We define an assending sequence $\mathscr{T}, \mathscr{T}^2, \cdots\cdots, \mathscr{T}^\sigma, \cdots\cdots$as rollows:

$$\mathscr{T}^\sigma = \begin{cases} \mathscr{T}.\ \mathscr{T}^{\sigma-1} & \text{if } \sigma \text{ is not a limit ordinal,} \\ \{G\mid G = \bigcup_{\alpha<\sigma}\mathscr{T}^\alpha(G) & \text{if } \sigma \text{ is a limit ordinal.}\end{cases}$$

It is easy to show that $\mathscr{T}^\sigma$ is a $1'2$-class for each ordinal $\sigma$. Define

$$\mathscr{T}^* = \bigcup_\sigma \mathscr{T}^\sigma$$

Then we have

PROPOSITION 2. 5. Let $\mathscr{R}$ be a $1'2$-class. Then $\mathscr{R}^*$ is a complete $1'2$-class. It is the smallest complete $1'2$-class containing $\mathscr{R}$. So, $\mathscr{R}$ is complete if and only if $\mathscr{R} = \mathscr{R}^*$.

The proof of this proposition is similar to the proof of Theorem 1.6 of [7]. $\mathscr{R}^*$ is called the completion of $\mathscr{R}$. Similarly to Theorem 1.7 of [7] we have

LEMMA 2.6. Let $\mathscr{R}$ be a $1'2$-class and $G$ be an $l$-group. Then $\mathscr{R}^*(G) \subseteq \mathscr{R}(G)^{\perp\perp}$. That is, $\mathscr{R}^* \subseteq \mathscr{R}^{\perp\perp}$ and $T_{1'27} \subseteq T_{1'25}$.

From Proposition 4.4 of [18] we can also see that $T_{1'27} \subseteq T_{1'25}$.

Since polars are closed convex $l$-subgroups, $T_{1'267} = T_{1'27}$. From the above lemmas we get

THEOREM 2.7. For radical classes of $l$-groups we have the following relations:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & & T_{13'4} & & & & & & \\
 & & & & & & & & \cap| & & & & & & \\
 & & & & & & & & T_{12} & & & & & & \\
 & & & & & & & & \cap| & & & & & & \\
T_{13'4} & \subseteq & T_{1'26} & \subseteq & T_{1'23} & \subseteq & T_{1'23'} & \subseteq & T_{1'2} & \supseteq & T_{1'24'} & \supseteq & T_{1'24} & \supseteq & T_{1'23'4} \\
 & & \cup| & & \cup| & & \cup| & & \cup| & & \cup| & & \cup| & & \\
 & & T_{1'256} & \subseteq & T_{1'235} & \subseteq & T_{1'23'5} & \subseteq & T_{1'25} & \supseteq & T_{1'24'5} & \supseteq & T_{1'245} & & \\
 & & \cup| & & \cup| & & \cup| & & \cup| & & & & & & \\
 & & T_{1'267} & = & T_{1'237} & = & T_{1'237} & = & T_{1'27} & & & & & &
\end{array}
$$

COROLLARY 2.8. Any polar kernel radical class is a product radical class and a subproduct radical class.

EXAMPLE 2.9. $\mathscr{F}_0$, the class of orthofinite $l$-groups, that is $l$-gruops in which no positive element exceeds an infinite pairwise disjoint set. We can show that $\mathscr{F}_0$ is a $1'25$-class. Suppose $G \in \mathscr{F}_0 \cdot \mathscr{F}_0$, that is $G/\mathscr{F}_0(G) \in \mathscr{F}_0$. Let $\{x_\alpha \mid \alpha \in A\}$ be a pairwise disjoint set of positive elements of $G$ with an upper bound $a$. Then $A = A_1 \bigcup A_2$, $A_1 \bigcap A_2 = \varnothing$ so that $x_{\alpha_1} \in \mathscr{F}_0(G)$ for $\alpha_1 \in A_1$ and $x_{\alpha_2} \overline{\in} \mathscr{F}_0(G)$ for $\alpha_2 \in A_2$. $\mathscr{F}_0(G) \in \mathscr{F}_0$ implies $|A_1|$ is finite. Then we have $[\mathscr{F}_0(G) + x_\alpha] \wedge [\mathscr{F}_0(G) + x_{\alpha'}] = \mathscr{F}_0(G) + x_\alpha \wedge x_{\alpha'} = \mathscr{F}_0(G)$ for $\alpha, \alpha' \in A_2$, $\alpha \neq \alpha'$. So $\{\mathscr{F}_0(G) + x_\alpha \mid \alpha \in A_2\}$ is a pairwise disjoint set of positive elements of $G/\mathscr{F}_0(G)$ with an upper bound $\mathscr{F}_0(G) + a$. Hence $|A_2|$ is also finite. Therefore $\mathscr{F}_0$ is a complete $1'2$-class. But $\mathscr{F}_0$ is not a $1'23'$-class.

EXAMPLE 2.10. $\mathscr{C}$, the class of all compolete $l$-groups, is a $1'23$-class, but not a $1'$ $23$-class.

EXAMPLE 2.11. Let $\mathscr{N}$ be the variety of normal-valued $l$-groups. Then $\mathscr{N} \in T_{1'256}$, but $\mathscr{N} \overline{\in} T_{1'27}$ by Proposition 4.6 of [18].

## 3. RADICAL MAPPINGS

Let $\mathscr{R}$ be a $1'2$-class and $G$ be an $l$-group. Let $\mathscr{R}(G)$ be the $\mathscr{R}$-radical of $G$. The mapping $G \to \mathscr{R}$ is called the radical mapping on $l$-groups which has the property: if $A$ is a convex $l$-subgroup of $G$, then $\mathscr{R}(A) = A \bigcap \mathscr{R}(G)$. Conversely, any mapping $\varphi$ associating to each $l$-group $G$ an $l$-ideal $\varphi(G)$ of $G$ and satistying the above property always define a unique radical class $\mathscr{R}$ such that $\mathscr{R}(G) = \varphi(G)$ for each $l$-group $G$ [10]. So a radical class is determined by its radical mapping. The above property is called the characteristic property of a radical mapping. In [7] J. Martinez gave the characteristic properties for torsion radical mapping. In [20] we gave the characteristic properties for product radical mappings as follows.

THEOREM 3.1 (Theorem 2.1 of [20]). A product radical calss $\mathscr{R}$ is uniquely determined by a product radical mapping $G \to \mathscr{R}(G)$ which has the characteristic properties: (I) if $A$ is a convex $l$-subgroup of $G$ then $\mathscr{R}(A) = A \bigcap \mathscr{R}(G)$; (II) if $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of $l$-groups, then $\mathscr{R}(\prod_{\lambda \in \Lambda} G_\lambda) = \prod_{\lambda \in \Lambda} \mathscr{R}(G_\lambda)$.

In this section we will prove the characteristic propertices for other radical mappings.

THEOREM 3. 2 A subproduct radical class $\mathscr{R}$ is uniquely determined by a subproduct radical mapping $G \to \mathscr{R}(G)$ which has the characteristic properties: ( I ) if $A$ is a convex $l$-subgroup of $G$ then $\mathscr{R}(A) = A \cap \mathscr{R}(G)$; ( II ) if $G$ is a completely subdirect product of $l$-groups $\{G_\lambda \mid \lambda \in \Lambda\}$ then $\mathscr{R}(G) = G \cap \prod_{\lambda \in \Lambda} \mathscr{R}(G_\lambda)$.

Proof. We only prove that the mappong $G \to \mathscr{R}(G)$ satisfies the property ( II ). The other parts of proof are similar to the proof of Theorem 2. 1 of [20]. Let $G$ be a completely subdirect product of $l$-groups $\{G_\lambda \mid \lambda \in \Lambda\}$. Put $\overline{G}_\lambda = \{g \in \prod_{\lambda \in \Lambda} G_\lambda \mid g_{\lambda'} = 0$ for $\lambda' \neq \lambda\}$ for each $\lambda \in \Lambda$. Next, for each $\lambda \in \Lambda$ and $x_\lambda \in G_\lambda$ we denote by $\bar{x}_\lambda$ the element of $G$ whose $\lambda$-coordinate is $x_\lambda$ and other coordinates are $O$. Then the mapping $\varphi: x_\lambda \to \bar{x}_\lambda$ is an isomorphism of $G_\lambda$ onto $\overline{G}_\lambda$. Hence $\varphi(\mathscr{R}(G_\lambda)) = \mathscr{R}(\overline{G}_\lambda)$.

a) For each $\lambda \in \Lambda$, $\mathscr{R}(G_\lambda)$ belongs to $\mathscr{R}$. Put $H = G \cap \prod_{\lambda \in \Lambda} \mathscr{R}(G_\lambda)$. Since $H$ is a completely subdirect product of the system $\{\mathscr{R}(G_\lambda) \mid \lambda \in \Lambda\}$, we obtain that $H \in \mathscr{R}$. Thus $H \subseteq \mathscr{R}(G)$.

b) For proving that $\mathscr{R}(G) \subseteq H$ it suffices to verify that $\mathscr{R}(G)^+ \subseteq H^+$. Let $x \in \mathscr{R}(G)^+$. For each $\lambda \in \Lambda$ let $x_\lambda$ be the coordinate of $x$ in $G_\lambda$. By way of contradiction, suppose that $x \overline{\in} H$. Hence there is $\lambda \in \Lambda$ with $x_\lambda \overline{\in} \mathscr{R}(G_\lambda)$. In view of the isomorphism $\varphi$, $\bar{x}_\lambda \overline{\in} \mathscr{R}(\overline{G})$. But $\mathscr{R}(\overline{G}_\lambda) = \mathscr{R}(G) \cap \overline{G}_\lambda$, hence $\bar{x}_\lambda \overline{\in} \mathscr{R}(G)$. We have $o \leqslant \bar{x}_s \leqslant x$ and this implies that $\bar{x}_\lambda \in \mathscr{R}(G)$, which is a contradiction.

The proof of the following theorem is left to the reader.

THEOREM 3. 3 $A$ complete radical class $\mathscr{R}_{1'25}$ is uniquely determined by a complete radical mapping $G \to \mathscr{R}_{1'25}(G)$ which has the characteristic properties: ( I ) if $A$ is a convex $l$-subgroup of $G$ then $\mathscr{R}_{1'25}(A) = A \cap \mathscr{R}_{1'25}(G)$; ( II ) for any $l$-group $G$ $\mathscr{R}_{1'25}(G/\mathscr{R}_{1'25}(G)) = 0$.

Form Theorem 3. 1, Theorem 3. 2 and Theorem 3. 3 we get the following theorems.

THEOREM 3. 4. $A$ complete product radical class $\mathscr{R}_{1'23'5}$ is uniquely determined by a complete product radical mapping $G \to \mathscr{R}_{1'23'5}(G)$ which has the characteristic properties: ( I ) if $A$ is a convex $l$-subgroup of $G$ then $\mathscr{R}_{1'23'5}(A) = A \cap \mathscr{R}_{1'23'5}(G)$, ( II ) if $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of $l$-groups then $\mathscr{R}_{1'23'5}(\prod_{\lambda \in \Lambda} G_\lambda) = \prod_{\lambda \in \Lambda} \mathscr{R}_{1'23'5}(G_\lambda)$, ( III ) for any $l$-group $G$ $\mathscr{R}_{1'23'5}(G/\mathscr{R}_{1'23'5}(G)) = 0$.

THEOREM 3. 5. A complete subproduct radical class $\mathscr{R}_{1'235}$ is uniquely determined by a complete subproduct radical mapping $G \to \mathscr{R}_{1'235}(G)$ which has the characteristic properties: ( I ) if A is a convex $l$-subgroup of $G$ then $\mathscr{R}_{1'235}(A) = A \cap \mathscr{R}_{1'235}(G)$; ( II ) if $G$ is a completely subdirect product of $\{G_\lambda \mid \lambda \in \Lambda\}$ then $\mathscr{R}_{1'235}(G) = G \cap \prod_{\lambda \in \Lambda} \mathscr{R}_{1'235}(G_\lambda)$; ( III ) for any $l$-group $G$ $\mathscr{R}_{1'235}(G/\mathscr{R}_{1'235}(G)) = 0$.

## 4. THE STRUCTURE OF A COMPLETE $l$-GROUP AND ARCHIMEDEAN $l$-GROUP

In order to give concrete discriptions of some important radical classes we need to know the structure of a complete $l$-group and an Archimedean $l$-group. First we introduce some concepts. Let $G$ be an $l$-group. We denote by v$G$ the least cardinal $\alpha$ such that $|A| \leqslant \alpha$ for each bounded disjoint subset $A$ of $G$, where $|A|$ denotes the cardinal of $A$. $G$ is said to be v-homogeneous of v$H$=v$G$ for any convex $l$-subgroup $H \neq \{0\}$ of $G$. $G$ is said to be v-homogeneous $l$-group of $\alpha$ type if v$G = \alpha$. An $l$-group $G$ is said to be continuous, if for any $0 < x \in G$ we have $x = x_1 + x_2$ and $x_1 \wedge x_2 = 0$, where $x_1 \neq 0$, $x_2 \neq 0$. By Theorem 3. 7 of [21] it is easy to verity the following lemma, the proof is left to the reader.

LEMMA 4. 1. Any complete $l$-group is $l$-isomorphic to an ideal subdirect product of complete v-homogeneous $l$-groups.

By using 4. 3 of [21] it is easy to verity that if an $l$-group $G$ is v-homogeneous and non-totally ordered, then v $G \geqslant \aleph_0$. It is well known that any non-zero complete totally ordered group is $l$-isomorphic to a real group R or an integer group Z. So from Lemma 4. 1 we obtain the structure of a complete $l$-group.

THEOREM 4. 2. Any complete $l$-group $G$ is $l$-isomorphic to an ideal subdirect product of real groups, integer groups and complete v-homogeneous $l$-groups of $\aleph_i$ type $(i \geqslant 0)$.

LEMMA 4. 3. (Proposition 2. 3 (1) of [22]) Let $G$ be a v-homogeneous $l$-group of $\aleph_i$ type and $G \neq \{0\}$. Then $G$ has no basic element.

LEMMA 4. 4. (Lemma 2. 4 of [22]) A complete $l$-group $G$ is continuous if and only if $G$ has no basic element.

COROLLARY 4. 5. A complete v-homogeneous $l$-group of $\aleph_i$ type is continuous.

Now we turn to an Archimedean $l$-group.

A subset $D$ in a lattice $L$ is called a d-set if there exists $x \in L$ such that $d_1 \wedge d_2 = x$ for any pair of distinct elements of $D$ and $d > x$ for each $d \in D$. We denote by $w\,[a, b]$ the least cardinal $\alpha$ such that $|D| \leqslant a$ for each d-set $D$ of $[a, b]$.

LEMMA 4. 6. An $l$-group $G$ is Archimedean if and only if $G$ is $l$-isomorphic to a subdirect product of subgroups of reals and Archimedean v-honogeneous $l$-groups of $\aleph_i$ type.

Proof. The sufficiency is clear. We need only to show the necessity.

Let $G$ be an Archimedean $l$-group. Then $G$ has the Dedekind completion $G^\wedge$. From Theorem 4. 2, without loss of generality, we have

$$\sum_{\delta \in \triangle} T_\delta \subseteq G^\wedge \subseteq^* \prod_{\delta \in \triangle} T_\delta, \tag{4. 1}$$

where $T_\delta = \mathrm{R}$ or Z or a continuous complete v-homogeneous $l$-group of $\aleph_i$ type for each $\delta \in \triangle$. Let $\rho_\delta$ be the projection map from $G^\wedge$ onto $T_\delta$. Put $\rho_\delta T_\delta = T_\delta'$,

$$\triangle_1 = \{\delta \in \triangle \mid T_\delta = \mathrm{R}\}, \triangle_2 = \{\delta \in \triangle \mid T_\delta = \mathrm{Z}\} \textit{ and } \triangle_3 = \triangle \backslash (\triangle_1 \cup \triangle_2).$$

Thus, for $\delta \in \triangle_1 \cup \triangle_2$ $T_\delta'$ is a subgroup of reals. For $\delta \in \triangle_3$ we can show that $T_\delta'$ is also v-homogeneous. In fact, for any $a, b \in T_\delta'$ $(a < b)$, we denote by $[a, b]^{T_\delta'}$ the interval in $T_\delta'$ and by $[a, b]^{T_\delta}$ the interval in $T_\delta$. We assume that $w\,[a, b]^{T_\delta} = \aleph_i$. $[a, b]^{T_\delta'} \subseteq [a, b]^{T_\delta}$ implies $w\,[a, b]^{T_\delta'} \leqslant w\,[a, b]^{T_\delta} = \aleph_i$. On the other hand, let $\{c_j \mid j \in J, |J| = \aleph_i\}$ be a disjoint subest in $[0, b-a]^{T_\delta}$. Since G is dense in $G^\wedge$, $T_\delta'$ is also dense in $T_\delta$. For each $c_j$ $(j \in J)$, there exists $0 < c_j' \in T_\delta'$ such that $c_j' \leqslant c_j$. Thus $\{c_j' \mid j \in J\}$ is also a disjoint subset in $[0, b-a]^{T_\delta'}$. So $w\,[a, b]^{T_\delta'} = w\,[0, b-a]^{T_\delta'} \geqslant \aleph_i$. Therefore $w\,[a, b]^{T_\delta'} = \aleph_i$ for any $a, b \in T_\delta'$, and so $T_\delta'$ is w-homogeneous. From 3. 6 in [21] $T_\delta'$ is v-homogenous. Since $T_\delta$ is complete, $T_\delta'$ is Archimedean. From (4. 1) we have

$$G \subseteq' \prod_{\delta \in \triangle} T_\delta',$$

where each $T_\delta'$ is a subgroup of reals or an Archimedean v-homogeneous $l$-group of $\aleph_i$ type for $\delta \in \triangle$.

Suppose that $G$ is a subdirect product of subgroups of reals and v-homogeneous $l$-groups of $\aleph_i$ type, $G \subseteq' \prod_{\delta \in \triangle} T_\delta$. Let $\triangle_1 = \{\delta \in \triangle \mid T_\delta$ is a subgroup of reals$\}$. If $\sum_{\delta \in \triangle_1} T_\delta \subseteq G$, $G$ is said to be a semicomplete subdirect product of subgroups of reals and v-homogeneous $l$-groups of $\aleph_i$ type, in symbols $\sum_{\delta \in \triangle_1 \subseteq \triangle} T_\delta \subseteq G \subseteq' \prod_{\delta \in \triangle} T_\delta$.

THEOREM 4. 7. An $l$-group $G$ is Archimedean if and only if $G$ is $l$-isomorphic to a semicomplete subdirect product of subgroups of reals and Archimedean v-homogeneous $l$-groups of $\aleph_1$ type.

Proof. We need only to show the necessity. By Lemma 4. 6, without loss of generality, we have

$$G \subseteq ' \prod_{\delta\in\triangle} T_\delta',$$

where each $T_\delta'$ is a subgroup of reals or an Archimedean v-homogeneous $l$-group of $\aleph_1$ type. Put $\triangle_1 = \{\delta\in\triangle \mid T_\delta'$ is a subgroup of reals$\}$. For each $\delta\in\triangle_1$ and any $0<t_\delta\in T_\delta'$ there exsits $0<x\in G$ such that $x_\delta=t_\delta$. Let $\bar{t}_\delta = (0, \cdots, 0, t_\delta, 0, \cdots, 0)$ be the element with only one non-zero component $t_\delta$. Since $\bar{t}_\delta\in G^{\hat{}}$ (see the formula (4. 1)) and $G$ is dense in $G^{\hat{}}$, there exists $\bar{t}_\delta' = (0, \cdots, 0, t_\delta', 0, \cdots, 0) \in G$ such that $t_\delta'\leqslant t_\delta$. Because $T_\delta'$ is a subgroup of reals, there exists some $n\in N$ such that $t_\delta<nt_\delta'$. Then $x\wedge n\,\bar{t}_\delta' = \bar{t}_\delta\in G$. Hence $T_\delta'\cong\bar{T}_\delta = \{\bar{t}_\delta \mid t_\delta\in T_\delta'\}\subseteq G$ for each $\delta\in\triangle_1$.
Therefore

$$\sum_{\delta\in\triangle_1\subseteq\triangle} T_\delta' \subseteq G \subseteq ' \prod_{\delta\in\triangle} T_\delta'.$$

## 5. THE RADICAL CLASSES GENERATED BY Z

For a family $X$ of $l$-groups we denote by $\mathscr{R}_{1'2i_3\cdots i_k}(X)$ the interesction of all $1'2i_3\cdots\cdots i_k$-classes containing $X$ where $i_3, \cdots, i_k\in\{3, 3', 4, 4', 5, 6, 7\}$. It is the smallest $1'2i_3\cdots\cdots i_k$-class containing $X$ and said to be the $1'2i_3\cdots\cdots i_k$-class generated by $X$. The $1'2i_3\cdots\cdots i_k$-class generated by a single $l$-group $G$ is denoted by $\mathscr{R}_{1'2i_3\cdots i_kG}$. It is well know that $\mathscr{R}_{13'4Z}=\mathscr{A}$, the variety of all abelian $l$-groups. In this section we will determine some radical classes generated by the integer group $Z$.

We recall that an element $g>0$ in an $l$-group $G$ is singular if $g=g_1+g_2$ with $g_1, g_2>0$ only when $g_1\wedge g_2=0$. A negative element $g$ is called a negative singular element if $-g$ is a singular element. $\mathscr{C}(G)$ will be denoted the set of all convex $l$-subgroups of an $l$-group $G$.

LEMMA 5. 1. An $l$-group G is a direct sum of Z if and only if $G$ is a complete $l$-group which has no continuous convex $l$-subgroup and each element of $G$ is a sum of singular elements and negative singular elements.

Proof. Let $G=\sum_{\alpha\in A} Z_\alpha$, $Z_\alpha=Z$ for all $\alpha\in A$. By Theorem 4. 2 $G$ is complete. Since Z is not continuous and every integer is a sum of singular elements 1 and negative singular elements $-1$, $G$ has no continuous convex $l$-subgroup and each element of $G$ is a sum of singular element and negative singular elements. Conversely, if $G$ is a complete $l$-group which has no continuous convex $l$-subgroup and each element of G is a sum of singular elements and negative singular elements. Since a complete v-homogeneous $l$-group of $\aleph_1$ type is continuous and the real group R has no singular element, it follows from Theorem 4. 2 that $G\subseteq^* \prod_{\alpha\in A} Z_\alpha$ with $Z_\alpha=Z$ for all $\alpha\in A$. But each element of $G$ is a sum of singular elements and negative singular elements, so $G=\sum_{\alpha\in A} Z_\alpha$.

THEOREM 5. 2. $\mathscr{R}_{1'2Z} = \{\sum_{\alpha\in A} Z_\alpha \mid Z_\alpha = Z$ for all $\alpha\in A\}$.

Proof. First we prove that the set $\mathscr{R}$ of all direct sums of Z is a $1'2$-class. It is clear that $\mathscr{R}$ is closed under taking convex $l$-subgroups, because any convex $l$-subgroup of a direct sum of Z is still a direct sum of Z. Suppose that $G_\lambda\in\mathscr{C}(G)$ and $G_\lambda=\sum_{\alpha_\lambda\in A_\lambda} Z_{\alpha_\lambda}$ $(Z_{\alpha_\lambda}=Z)$ for $\lambda\in\Lambda$. It is well known that $\mathscr{C}$ of all complete $l$-groups is a radical class [13], that is $\mathscr{C}$ is closed under taking joins of convex $l$-subgroups. So $\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda$ is complete. $\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda$ has no continuous convex $l$-subgroup. In fact, if $H$ is a convex $l$-subgroup of $\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda$. Since $\mathscr{C}(G)$ is a Brouweian lattice,

$$H = H \cap (\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda) = \bigvee_{\lambda\in\Lambda}^{(G)} (H \cap G_\lambda).$$

Each $H \cap G_\lambda$ is a convex $l$-subgroup of $G_\lambda$, so $H \cap G_\lambda = \sum_{\alpha_\lambda' \in A_\lambda' \subseteq A_\lambda} Z_{\alpha_\lambda'} (Z_{\alpha_\lambda'} = Z)$. Hence for each $\lambda \in \Lambda$, if $0 < z_{\alpha_\lambda'} \in Z_{\alpha_\lambda'} \subseteq H$ $(\alpha_\lambda' \in A_\lambda')$, then $z_{\alpha_\lambda'}$ cannot be expressed to $z_{\alpha_\lambda'} = x_1 + x_2$ such that $x_1 \wedge x_2 = 0$ and $x_1 \neq 0$, $x_2 \neq 0$. So H is not continuous. Let $x \in \bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda$. Then $x = x_1 + \cdots\cdots + x_n$ with $x_i \in G_{\lambda_i}$. Since each $x_i$ is a sum of singular elements and negative singular elements, $x$ is a sum of singular elements and negative singular elements. Therefore $\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda$ is also a direct sum of Z by Lemma 5. 1.

Now suppose that $\mathscr{U}$ is a $1'2$-class containing Z. Let $\sum_{\alpha\in A} Z_\alpha$ $(Z_\alpha = Z)$ be a direct sum of Z. Since $\mathscr{U}$ is closed under taking joins of convex $l$-subgroups and

$$\bigvee_{\alpha\in A}^{(\Pi_{\alpha\in A} Z_\alpha)} Z_\alpha = \sum_{\alpha\in A} Z_\alpha$$

by Corollary 1 of Theorem 1. 5 in [15], $\sum_{\alpha\in A} Z_\alpha \in \mathscr{U}$. This shows that $\mathscr{R}$ is the smallest $1'2$-class containing Z.

LEMMA 5. 3. An $l$-group $G$ is an ideal subdirect product of Z if and only if $G$ is a complete $l$-group which has no continuous convex $l$-subgroup and each convex $l$-subgroup of $G$ has a singular element.

Proof. The necessity is clear. Suppose that $G$ is a complete $l$-group which has no continuous convex $l$-subgroup and each convex $l$-subgroup of G has a singular element. By Theorem 4. 2 we have

$$G \subseteq^* \prod_{\lambda\in\Lambda} G_\lambda$$

where each $G_\lambda$ is Z or R or a complete $v$-homogeneous $l$-group of $\aleph_i$ type. Since a complete $v$-homogeneous $l$-group of $\aleph_i$ type is continuous and R has no singulat element, so

$$G \subseteq^* \prod_{\lambda\in\Lambda} Z_\lambda$$

where $Z_\lambda = Z$ for each $\lambda \in \Lambda$.

THEOREM 5. 4. $\mathscr{R}_{1'23'Z} = \{G \mid G \subseteq^* \prod_{\alpha\in A} Z_\alpha,\ Z_\alpha = Z \text{ for all } \alpha \in A\}$.

Proof. First we prove that the set $\mathscr{R}$ of all ideal subdirect products of Z is a $1'23'$-class. $\mathscr{R}$ is closed under taking convex $l$-subgroups, because any convex $l$-subgroup of an ideal subdirect product of Z is still an ideal subdirect product of Z. Suppose that $G$ is an $l$-group and $G_\lambda \in \mathscr{C}(G)$,

$$G_\lambda \subseteq^* \prod_{\alpha_\lambda \in A_\lambda} Z_{\alpha_\lambda} (Z_{\alpha_\lambda} = Z)$$

for $\lambda \in \Lambda$. Similarly to the proof of Theorem 5. 2 we see that $\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda$ is complete and has no continuous convex $l$-subgroup. Let $H$ be a convex $l$-subgroup of $\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda$, then

$$H = \bigvee_{\lambda\in\Lambda}^{(G)} (H \cap G_\lambda).$$

For each $\lambda \in \Lambda$, $H \cap G_\lambda$ is a convex $l$-subgroup of $G_\lambda$, so $H \cap G_\lambda \subseteq^* \prod_{\alpha_\lambda \in A_\lambda} Z_{\alpha_\lambda} (Z_{\alpha_\lambda} = Z)$. Hence $H$ has a singular element. It follows from Lemma 5. 3 that $\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda \in \mathscr{R}$.

Now suppose that $\mathscr{U}$ is a $1'23'$-class containing Z. Since a convex $l$-subgroup of direct product of Z is an ideal subirect product of Z, so $\mathscr{U} \supseteq \mathscr{R}$ and $\mathscr{R}$ is the smallest $1'23'$-class containing Z.

LEMMA 5. 5. An $l$-group $G$ is a completely subdirect product of Z if and only if $G$ is an Archimedean $l$-group which has no continuous convex $l$-subgroup and each convex $l$-subgroup of $G$ has a singular element.

Proof. Necessity. Let $\sum_{\alpha\in A} Z_\alpha \subseteq G \subseteq' \prod_{\alpha\in A} Z_\alpha$ ($Z_\alpha = Z$). By Theorem 4. 7 $G$ is Archimedean. It is clear that $G$ has no continuous convex $l$-subgroup. Each convex $l$-subgroup $H$ of $G$ contains at least a $Z_\alpha$, so $H$ has a singular element.

Sufficiency. Suppose that $G$ is an Archimedean $l$-group which has no continuous convex $l$-subgroup and each convex $l$-subgroup of $G$ has a singular element. By Theorem 4. 7 we ha

$$\sum_{\lambda\in\Lambda} G_\lambda \subseteq G \subseteq' \prod_{\lambda\in\Lambda} G_\lambda$$

where each $G_\lambda$ is Z or R. But each $G_\lambda$ is a convex $l$-subgroup of $G$ and R has no singular element, so $\sum_{\lambda\in\Lambda} Z_\lambda \subseteq G \subseteq' \prod_{\lambda\in\Lambda} Z_\lambda$ ($Z_\lambda = Z$).

THEOREM 5. 6. $\mathscr{R}_{1'23Z} = \{G \mid \sum_{\alpha\in A} Z_\alpha \subseteq G \subseteq' \prod_{\alpha\in A} Z_\alpha,\ Z_\alpha = Z \text{ for all } \alpha\in A\}$.

Proof. First we prove that the set $\mathscr{R}$ of all complete subdirect products of Z is a $1'23$-class. $\mathscr{R}$ is closed under taking convex $l$-subgroups, because any convex $l$-subgroup of a completely subdirect prodeut of Z is still a completely subdirect product of Z. Suppose that $G$ is an $l$-group and $G_\lambda \in \mathscr{C}(G)$, $G_\lambda \in \mathscr{R}$ ($\lambda\in\Lambda$). Since $\mathscr{A}_I$ , the set of all Archimedean $l$-groups, is a quasi-torsion class [14] and is closed under taking joins of convex $l$-subgroups. So $\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda$ is Archimedean. Similarly to the proof of Theorem 5. 2 and Theorem 5. 4 we see that $\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda$ has no continuous convex $l$-subgroup and each convex $l$-subgroup of $G$ has a singular element. It follows from Lemma 5. 5 that $\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda \in \mathscr{R}$. It is obvious that $\mathscr{R}$ is the smallest $1'23$-class containing Z.

The following proposition is a corollary of Theorem 2. 7.

PROPOSITION 5. 7. Let $G$ be an $l$-group, then we have the following relationship between the radical classes generated by $G$:

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & \mathscr{R}_{13'4G} & & & & & & & & \\
 & & & & & & & & \cup\!| & & & & & & & & \\
 & & & & & & & & \mathscr{R}_{12G} & & & & & & & & \\
 & & & & & & & & \cup\!| & & & & & & & & \\
\mathscr{R}_{13'4G} & \supseteq & \mathscr{R}_{1'26G} & \supseteq & \mathscr{R}_{1'23G} & \supseteq & \mathscr{R}_{1'23'G} & \supseteq & \mathscr{R}_{1'2G} & \subseteq & \mathscr{R}_{1'24'G} & \subseteq & \mathscr{R}_{1'24G} & \subseteq & \mathscr{R}_{13'4G} \\
 & & \cap\!| & & \cap\!| & & \cap\!| & & \cap\!| & & \cap\!| & & \cap\!| & & \\
 & & \mathscr{R}_{1'256G} & \supseteq & \mathscr{R}_{1'235G} & \supseteq & \mathscr{R}_{1'23'5G} & \supseteq & \mathscr{R}_{1'25G} & \subseteq & \mathscr{R}_{1'24'5G} & \subseteq & \mathscr{R}_{1'245G} & & \\
 & & \cap\!| & & \cap\!| & & \cap\!| & & \cap\!| & & & & & & \\
 & & \mathscr{R}_{1'267G} & = & \mathscr{R}_{1'237G} & = & \mathscr{R}_{1'23'7G} & = & \mathscr{R}_{1'27} & & & & & &
\end{array}$$

Let $\mathscr{R}$ be a radical class. In [18] M. Darnel defined the order closure $\mathscr{R}^c$ of $\mathscr{R}$ with $\mathscr{R}^c(G) = \overline{\mathscr{R}(G)_G}$ for any $l$-group $G$. It follows from Lemma 2. 2 and Proposition 5. 7 that

$$\mathscr{R}_{1'26Z} = \mathscr{R}^c_{1'2Z} = \mathscr{R}^c_{1'23'Z} = \mathscr{R}^c_{1'23Z}. \tag{5.1}$$

From theorem 5. 2, Theorem 5. 4, Theorem 5. 6 and the formula (5. 1) we get

THEOREM 5. 8. ( I ) $\mathscr{R}_{1'26Z} = \{G \mid G$ is an order closure of a convex $l$-subgroup $\sum_{\alpha\in A} Z_\alpha$ ($Z_\alpha = Z$) of an $l$-group $H\}$.

( II ) $\mathscr{R}_{1'26Z} = \{G \mid G$ is an order closure of a convex $l$-subgroup $K$ of an $l$-group $H$ where $K \subseteq^* \prod_{\alpha\in A} Z_\alpha$ ($Z_\alpha = Z$)$\}$.

( III ) $\mathscr{R}_{1'26Z} = \{G \mid G$ is an order closure of a convex $l$-subgroup $K$ of an $l$-group $H$ where $\sum_{\alpha\in A} Z_\alpha \subseteq K \subseteq' \prod_{\alpha\in A} Z_\alpha$ ($Z_\alpha = Z$)$\}$.

From Lemma 2.4, Theorem 5.2, Theorem 5.4, Theorem 5. 6 and Proposition 5.7 we have

THEOREM 5.9. ( I ) $\mathscr{R}_{1'27Z}=\{G \mid G$ is a double polar of a a convex $l$-subgtoup $\sum_{\alpha\in A} Z_\alpha$ $(Z_\alpha=Z)$ of an $l$-group $H\}$.

( II ) $\mathscr{R}_{1'27Z}=\mathscr{R}_{1'23'7Z}=\{G \mid G$ is a double polar of a convex $l$-subgroup $K$ of an $l$-group $H$ where $K\subseteq^{*} \prod_{\alpha\in A} Z_\alpha$ $(Z_\alpha=Z)\}$.

( III ) $\mathscr{R}_{1'27Z}=\mathscr{R}_{1'237Z}=\{G \mid G$ is a double polar of a convex $l$-subgroup $K$ of an $l$-group $H$ where $\sum_{\alpha\in A} Z_\alpha \subseteq K \subseteq' \prod_{\alpha\in A} Z_\alpha$ $(Z_\alpha=Z)\}$.

LEMMA 5.10. Let $\mathscr{R}$ be a radical class, then $\mathscr{R}^{\perp\perp}=\{G \mid$ for each convex $l$-subgroup $C$ of $G$ $\mathscr{R}(C)\neq 0\}$.

Proof. $\mathscr{R}^{\perp}(G)$ is the largest convex $l$-subgroup $C$ of $G$ such that $\mathscr{R}(C)=0$. So $\mathscr{R}^{\perp}(G)=0$ if and only if for each convex $l$-subgroup $C$ of $G$ $\mathscr{R}(C)\neq 0$. Since $\mathscr{R}^{\perp}(G)=\mathscr{R}(G)^{\perp}$, $G\in\mathscr{R}^{\perp\perp}$ if and only if $\mathscr{R}^{\perp}(G)=0$, if and only if for each convex $l$-subgroup $C$ of $G$ $\mathscr{R}(C)\neq 0$.

Let $\mathscr{R}$ be a radical class. It is clear that $\mathscr{R}^{\perp\perp}$ is the smallest polar radical class containing $\mathscr{R}$. From Lemma 2.4 and Lemma 5.10 we get

THEOREM 5.11. $\mathscr{R}_{1'27Z}=\mathscr{R}_{1'2Z}^{\perp\perp}=\{G \mid$ each convex $l$-subgroup of $G$ contains a convex $l$-subgroup $\sum_{\alpha\in A} Z_\alpha$ $(Z_\alpha=Z)\}$.

## 6. THE RADICAL CLASSES GENERATED BY R

In this section we will determine some radical classes generated by the real group R.

LEMMA 6.1. An $l$-group $G$ is a direct sum of R if and only if $G$ is a complete $l$-group which has no continuous convex $l$-subgroup and for each principal convex $l$-subgroup $C$ of $G$ $vC$ is finite and $|C|>\aleph_0$.

Proof. Let $G=\sum_{\alpha\in A} R_\alpha (R_\alpha=R)$. By Theorem 4.2 $G$ is complete. Since R is not continuous and R is a totally ordered group, $G$ has no continuous convex $l$-subgroup and $|K|>\aleph_0$ for each convex $l$-subgroup $K$ of $G$. Since each element of $G$ has only finitely many non-zero components, $v\,C$ is finite for each principal convex $l$-subgroup $C$ of $G$.

Conversely, suppose that $G$ satisfies the conditions of Lemma 6.1. Since a complete $v$-homogeneous $l$-group of $\aleph_i$ type is continuous and $|Z|=\aleph_0$, $G\subseteq \prod_{\alpha\in A} R_\alpha$ $(R_\alpha=R)$ by Theorem 4.2. The fact that $v\,C$ is finite for each principal convex $l$-subgroup $C$ of $G$ implies that each element of $G$ has only finitely many non-zero components. Therefore $G=\sum_{\alpha\in A} R_\alpha$ $(R_\alpha=R)$.

THEOREM 6.2. $\mathscr{R}_{1'2R}=\{\sum_{\alpha\in A} R_\alpha \mid R_\alpha=R$ for all $\alpha\in A\}$.

Proof. We can prove that the set $\mathscr{R}$ of all direct sums of R is a $1'2$-class. It is clear that $\mathscr{R}$ is closed under taking convex $l$-subgroups. Suppose that $G$ is an $l$-group and $G_\lambda\in\mathscr{C}(G)$, $G_\lambda=\sum_{\alpha_\lambda\in A_\lambda} R_{\alpha_\lambda}$ $(R_{\alpha_\lambda}=R)$ for $\lambda\in\Lambda$. Similarly to the proof of Theorem 5.2 we can show that $\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda$ is complete and has no continuous convex $l$-subgroup.

Now we prove that $v\,C$ is finite for each principal convex $l$-subgroup $C$ of $\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda$, Let $0<x\in\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda$. Then

$$x=x_{\lambda_1}+\cdots\cdots+x_{\lambda_n}\leqslant x_{\lambda_1}^{+}+\cdots\cdots+x_{\lambda_n}^{+},$$

where $x_{\lambda_i}\in G_{\lambda_i}$ $(1\leqslant i\leqslant n)$. Let $G_0$ be the convex $l$-subgroup generated by $x$ in $\bigvee_{\lambda\in\Lambda}^{(G)} G_\lambda$. Suppose that $\{x_\alpha | \alpha \in A\}$ is a disjoint subset of $G_0$. We assume $x_\alpha \leqslant x$ for each $\alpha \in A$ (otherwise let $\bar{x}_\alpha = x_\alpha \wedge x$). Put $x^i_\alpha = x_\alpha \wedge x^+_{\lambda_i}$, $i=1, \cdots\cdots, n$. For each $\alpha \in A$ there at least exists $x^i_\alpha \neq 0$. Because if all $x^i_\alpha = 0$ $(i=1, \cdots, n)$, then

$$0 < x_\alpha = x_\alpha \wedge x \leqslant x_\alpha \wedge (x^+_{\lambda_1} + \cdots\cdots + x^+_{\lambda_n}) \leqslant x_\alpha \wedge x^+_{\lambda_1} + \cdots\cdots + x_\alpha \wedge x^+_{\lambda_n} = 0,$$

a contradiction. It is clear that

$$x^i_\alpha \wedge x^i_{\alpha'} = x_\alpha \wedge x_{\alpha'} \wedge x^+_{\lambda_i} = 0 (\alpha \neq \alpha'),$$

and so $\{x^i_\alpha | \alpha \in A\}$ is a disjoint subset of $G_{\lambda_i}(x^+_{\lambda_i})$ for $i=1, \cdots, n$. Since $v\, G_{\lambda_i}(x^+_{\lambda_i})$ is finite for $i=1, \cdots, n$, $|A|$ must be finite. Combining the above we see that $\bigvee\limits_{\lambda \in A}{}^{(G)} G_\lambda = \sum\limits_{\alpha \in A} H_\alpha$ ($H_\alpha = \mathrm{Z}$ or $\mathrm{R}$) by Theorem 4. 2. Since any convex $l$-subgroup $K$ of $\sum\limits_{\alpha \in A} H_\alpha$ is also a join of direct sums of R and $|K| > \aleph_0$, $\sum\limits_{\alpha \in A} H_\alpha$ cannot contain Z as a convex $l$-subgroup. Hence $\bigvee\limits_{\lambda \in A}{}^{(G)} G_\lambda = \sum\limits_{\alpha \in A} \mathrm{R}_\alpha$ ($\mathrm{R}_\alpha = \mathrm{R}$) by Lemma 6. 1.

Similarly to the proof of Theorem 5. 2 we can show that $\mathscr{R}$ is the smallest 1′2-class containing R.

LEMMA 6. 3. An $l$-groap $G$ is an ideal subdirect product of R if and only of $G$ is a complete $l$-group which has no continuous convex $l$-subgroup and $|K| > \aleph_0$ for each convex $l$-subgroup $K$ of $G$.

The proof of this lemma is obvious by Theorem 4. 2.

THEOREM 6. 4. $\mathscr{R}_{1'23'\mathrm{R}} = \{G | G \subseteq^* \prod\limits_{\alpha \in A} \mathrm{R}_\alpha,\ \mathrm{R}_\alpha = \mathrm{R}$ for all $\alpha \in A\}$.

The proof of this theorem is similar to those of Theorem 5. 4 and Theorem 6. 2.

LEMMA 6. 5. An $l$-group $G$ is a completely subdirect product of R if and only if $G$ is an Archimedean $l$-group which has no continuous convex $l$-subgroup and $|K| > \aleph_0$ for each convex $l$-subgroup $K$ of $G$.

The proof of this lemma is obvious by Theorem 4. 7.

THEOREM 6. 6. $\mathscr{R}_{1'23\mathrm{R}} = \{G | \sum\limits_{\alpha \in A} \mathrm{R}_\alpha \subseteq G \subseteq' \prod\limits_{\alpha \in A} \mathrm{R}_\alpha,\ \mathrm{R}_\alpha = \mathrm{R}$ for all $\alpha \in A\}$.

The proof of this theorem is similar to those of Theorem 5. 6 and Theorem 6. 2.

Similarly to Theorem 5. 8 we have

THEOREM 6. 7. ( I ) $\mathscr{R}_{1'26\mathrm{R}} = \{G | G$ is an order closure of a convex $l$-subgroup $\sum\limits_{\alpha \in A} \mathrm{R}_\alpha$ ($\mathrm{R}_\alpha = \mathrm{R}$) of an $l$-group $H\}$.

( II ) $\mathscr{R}_{1'26\mathrm{R}} = \{G | G$ is an closure of a convex $l$-subgroup $K$ of an $l$-group $H$ where $K \subseteq^* \prod\limits_{\alpha \in A} \mathrm{R}_\alpha$ ($\mathrm{R}_\alpha = \mathrm{R}$)$\}$.

( III ) $\mathscr{R}_{1'26\mathrm{R}} = \{G | G$ is an order closure of a convex $l$-subgroup $K$ of an $l$-group $H$ where $\sum\limits_{\alpha \in A} \mathrm{R}_\alpha \subseteq K \subseteq' \prod\limits_{\alpha \in A} \mathrm{R}_\alpha$ ($\mathrm{R}_\alpha = \mathrm{R}$)$\}$.

Similary to Theorem 5. 9 we have

THEOREM 6. 8. ( I ) $\mathscr{R}_{1'27\mathrm{R}} = \{G | G$ is a double polar of a convex $l$-subgroup $\sum\limits_{\alpha \in A} \mathrm{R}_\alpha$ ($\mathrm{R}_\alpha = \mathrm{R}$) of an $l$-group $H\}$.

( II ) $\mathscr{R}_{1'27\mathrm{R}} = \mathscr{R}_{1'23'7\mathrm{R}} = \{G | G$ is a double polar of a convex $l$-subgroup $K$ of an $l$-group $H$ where $K \subseteq^* \prod\limits_{\alpha \in A} \mathrm{R}_\alpha$ ($\mathrm{R}_\alpha = \mathrm{R}$)$\}$.

( III ) $\mathscr{R}_{1'27\mathrm{R}} = \mathscr{R}_{1'237\mathrm{R}} = \{G | G$ is a double polar of a convex $l$-subgroup $K$ of an $l$-group $H$ where $\sum\limits_{\alpha \in A} \mathrm{R}_\alpha \subseteq \mathrm{K} \subseteq' \prod\limits_{\alpha \in A} \mathrm{R}_\alpha$ ($\mathrm{R}_\alpha = \mathrm{R}$)$\}$.

Similarly to Theorem 5. 11 we have

THEOREM 6. 9. $\mathscr{R}_{l'27R}=\mathscr{R}^{\perp\perp}_{l'2R}=$ $\{G \mid$ each convex $l$-subgroup of $G$ contains a convex $l$-subgroup $\sum\limits_{\alpha\in A} R_\alpha$ $(R_\alpha=R)\}$.

## 7. AN EXAMPLE

We consider the totall ordered group $Z\vec{\times}Z$. $Z_0=\{(0.\ z) \mid z\in Z\}\cong Z$ is an $l$-ideal of $Z\vec{\times}Z$. It is clear that $Z_0^{\perp\perp}=Z\vec{\times}Z$ and $Z\vec{\times}Z/Z_0\cong Z_0$. So $Z\vec{\times}Z\in\mathscr{R}_{l'27Z}$ and $Z\vec{\times}Z\in\mathscr{R}_{l'25Z}$, but $Z\vec{\times}Z\overline{\in}\mathscr{R}_{l'2Z}$. Hence $\mathscr{R}_{l'2Z}\neq\mathscr{R}_{l'25Z}$. $\mathscr{R}_{l'25Z}=\mathscr{R}^{*}_{l'2Z}$.

Similarly, $R\vec{\times}R\in\mathscr{R}_{l'25R}$ and $R\vec{\times}R\overline{\in}\mathscr{R}_{l'2R}$. Hence $\mathscr{R}_{l'2R}\neq\mathscr{R}_{l'25R}$, $\mathscr{R}_{l'25R}=\mathscr{R}^{*}_{l'2R}$.

Note. Since Z and R have no proper convex $l$-subgroup, $\mathscr{R}_{l'2Z}$ and $\mathscr{R}_{l'2R}$ are closed under $l$-homomorphisms. Therefore $\mathscr{R}_{l'24'Z}=\mathscr{R}_{l'24Z}=\mathscr{R}_{l'2Z}$ and $\mathscr{R}_{l'24'R}=\mathscr{R}_{l'24R}=\mathscr{R}_{l'2R}$.

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