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A RESULT OF COMMUTATIVITY OF RINGS

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Abstract. In this paper we prove the following:

THEOREM. Let $n > 1$ and m be fixed relatively prime positive integers and k is any non-negative integer. If R is a ring with unity 1 satisfying $x^k[x^n, y] = [x, y^m]$ for all $x, y \in R$ then R is commutative.

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1. INTRODUCTION.

Psomopoulos [12] proved that if R is a ring with unity satisfying the properties that for each $x, y \in R$,

(i) $x^k[x^n, y] = [x, y^m]$

(ii) $(xy)^n = x^n y^n$

(iii) $(xy)^k = x^k y^k$

where $n > 1$ and m are fixed relatively prime positive integers and k is any non-negative integer, then R is commutative. In this paper we prove the theorem stated in the abstract which improve above theorem of Psomopolous [12] where conditions (ii) and (iii) are superfluous.

Throughout, R will denote an associative ring with unit 1. We use the following notations.

$Z(R)$, the center of R.

$[x, y] = xy - yx$

$C(R)$, the commutator ideal of R.

$N(R)$, the set of all nilpotent elements of R.

$D(R)$, the set of all zero divisors in R.

2. MAIN RESULTS.

We state our main result as follows.

MAIN THEOREM. Let $n > 1$ and m be fixed relatively prime positive integers and k is any non-negative integer. If R is a ring with unity 1 satisfying

(*) $$x^k[x^n, y] = [x, y^m] \quad \text{for all} \quad x, y \in R$$

then R is commutative.

We begin with the following lemmas which will be used in proving our main theorem.

LEMMA 1 ([2], Theorem 1). Let R be a ring satisfying an identity $q(X) = 0$, where $q(X)$ is a polynomial identity in non-commuting in-determinates, its coefficient being integers with highest common factor one. If there exists no prime p for which the ring of 2×2 matrices over $GF(p)$ satisfies $q(X) = 0$, then R has a nil commutator ideal and the nilpotent elements of R form an ideal.

LEMMA 2 ([8], p. 221). If $x, y \in R$ and $[x, y]$ commute with x, then $[x^n, y] = nx^{n-1}[x, y]$ for all positive integer n.

LEMMA 3 ([9]). Let R be a ring with unity and let $f : R \to R$ be a function such that $f(x+1) = f(x)$ for all $x \in R$. If for some positive integer n, $x^n f(x) = 0$ for all x in R, then necessarily $f(x) = 0$.

LEMMA 4. If R is a ring satisfying (*) in the hypothesis of the main theorem then

$$C(R) \subseteq N(R) \subseteq Z(R)$$

PROOF. By Lemma 3 of [12] we have $N(R) \subseteq Z(R)$ when R satisfies $x^k[x^n, y] = [x, y^m]$ for all $x, y \in R$. This is a polynomial identity with coprime integral coefficients. But if we consider (i) $x = e_{22}$ and $y = e_{21}$, if $n > 1$, $m > 1$ and (ii) $x = e_{21}$ and $y = e_{22}$ if $n > 1$ and $m = 1$, we find that no ring of 2×2 matrices over $GF(p)$, p a prime, satisfies this identity. Hence by Lemma 1, $C(R)$ is a nil ideal and thus

$$C(R) \subseteq N(R) \subseteq Z(R).$$

PROOF OF MAIN THEOREM. By Lemma 4, we have

$$C(R) \subseteq N(R) \subseteq Z(R)$$

Thus all commutators are central. Moreover, we know that R is isomorphic to a subdirect sum of subdirectly irreducible rings R_α each of which a homomorphic image of R satisfies the hypotheses of the theorem. Thus we can assume that R is subdirectly irreducible ring. Hence I, the intersection of all non-zero ideals is non-zero.

CASE 1. Let $n > 1$ and $m > 1$.

By using Lemma 2, we write (*) as

$$nx^{n+k-1}[x, y] = [x, y^m] \quad \text{for all} \quad x, y \in R. \tag{2.1}$$

Let $c = 2^{n+k} - 2 > 0$, then

$$\begin{aligned} ncx^{n+k-1}[x, y] &= n\{2^{n+k}x^{n+k-1}[x, y] - 2x^{n+k-1}[x, y]\} \\ &= n2^{n+k}x^{n+k-1}[x, y] - 2nx^{n+k-1}[x, y] \\ &= n(2x)^{n+k-1}[2x, y] - 2[x, y^m] \\ &= [2x, y^m] - 2[x, y^m] = 0. \end{aligned} \tag{2.2}$$

Hence $ncx^{n+k-1}[x, y] = 0$ for all $x, y \in R$. Now replace x by $x + 1$ and by using Lemma 3, we get

$$nc[x, y] = 0. \tag{2.3}$$

All cummutators are central and hence by Lemma 2

$$[x^{nc}, y] = ncx^{nc-1}[x, y] = 0.$$

Thus $x^{nc} \in Z(R)$ for all $x \in R$. We replace y by y^m in (2.1) to get

$$nx^{n+k-1}[x, y^m] = [x, (y^m)^m]. \tag{2.4}$$

Thus

$$nx^{n+k-1}[x,y^m] = n[x,y^m]x^{n+k-1}$$
$$= nmy^{m-1}[x,y]x^{n+k-1}$$
$$= nmy^{m-1}x^{n+k-1}[x,y]$$
$$= my^{m-1}[x,y^m] \tag{2.5}$$

and

$$[x,(y^m)^m] = m(y^m)^{m-1}[x,y^m] = my^{m-1}y^{(m-1)^2}[x,y^m]. \tag{2.6}$$

Thus by using (2.5) and (2.6), we can write (2.4) as

$$my^{m-1}[x,y^m] = my^{m-1}y^{(m-1)^2}[x,y^m]$$
$$my^{m-1}\left(1-y^{(m-1)^2}\right)[x,y^m] = 0$$

Hence

$$my^{m-1}\left(1-y^{nc(m-1)^2}\right)[x,y^m] = 0. \tag{2.7}$$

We claim that

$$D(R) \subseteq Z(R).$$

Let $a \in D(R)$ then

$$a^{nc(m-1)^2} \in Z(R) \cap D(R) \quad \text{and} \quad Ia^{nc(m-1)^2} = 0.$$

By (2.7), we get

$$ma^{m-1}\left(1-a^{nc(m-1)^2}\right)[x,a^m] = 0.$$

Thus

$$\left(1-a^{nc(m-1)^2}\right)ma^{m-1}[x,a^m] = 0. \tag{2.8}$$

If $ma^{m-1}[x,a^m] \neq 0$, then

$$1-a^{nc(m-1)^2} \in D(R)$$

Hence $I(1-a^{nc(m-1)^2}) = 0$ and $I = 0$. This is contradiction. Now we have

$$ma^{m-1}[x,a^m] = 0. \tag{2.9}$$

Thus

$$n^2x^{n+k-1}x^{n+k-1}[x,a] = nx^{n+k-1}[x,a^m]$$
$$= [x,(a^m)^m]$$
$$= m(a^m)^{m-1}[x,a^m]$$
$$= a^{(m-1)^2}ma^{m-1}[x,a^m] = 0. \tag{2.10}$$

Replacing x by $x+1$ in (2.10) and using Lemma 3 we get

$$n^2[x,a] = 0. \tag{2.11}$$

By using Lemma 2, we can write (*) as

$$x^k[x^n,y] = my^{m-1}[x,y]. \tag{2.12}$$

Let $d = 2^m - 2 > 0$. Then

$$\begin{aligned} mdy^{m-1}[x,y] &= m2^m y^{m-1}[x,y] - 2y^{m-1}[x,y] \\ &= m(2y)^{m-1}[x,2y] - 2my^{m-1}[x,y] \\ &= x^k[x^n,2y] - 2x^k[x^n,y] \\ &= x^k[x^n,2y] - x^k[x^n,2y] = 0. \end{aligned} \tag{2.13}$$

Hence $mdy^{m-1}[x,y] = 0$ for all $x,y \in R$. Now replacing y by $y+1$ and by using Lemma 3, we get

$$md[x,y] = 0. \tag{2.14}$$

All commutators are central and hence by Lemma 2

$$[x,y^{md}] = mdy^{md-1}[x,y] = 0$$

Thus $y^{md} \in Z(R)$ for all $y \in R$. Now replacing x by x^n in (2.12), we get

$$x^{nk}[(x^n)^n,y] = my^{m-1}[x^n,y] \tag{2.15}$$

Thus

$$\begin{aligned} x^{nk}[(x^n)^n,y] &= x^{nk}n(x^n)^{n-1}[x^n,y] \\ &= nx^{nk}x^{n-1}x^{(n-1)^2}[x^n,y] \\ &= nx^{n+k-1}x^{nk-k}x^{(n-1)^2}[x^n,y] \\ &= nx^{nx^{n+k-1}}x^{(n-1)k}x^{(n-1)^2}[x^n,y] \\ &= nx^{n+k-1}x^{(n-1)(n+k-1)}[x^n,y] \end{aligned} \tag{2.16}$$

$$\begin{aligned} my^{m-1}[x^n,y] &= m[x^n,y]y^{m-1} \\ &= mnx^{n-1}[x,y]y^{m-1} \\ &= mnx^{n-1}y^{m-1}[x,y] \\ &= nx^{n-1}my^{m-1}[x,y] \\ &= nx^{m-1}x^k[x^n,y] \\ &= nx^{n+k-1}[x^n,y] \end{aligned} \tag{2.17}$$

Thus by using (2.16) and (2.17) we can write (2.15) as

$$nx^{n+k-1}x^{(n-1)(n+k-1)}[x^n,y] = nx^{n+k-1}[x^n,y].$$

$$nx^{n+k-1}(1-x^{(n-1)(n+k-1)})[x^n,y] = 0. \tag{2.18}$$

Hence by using (2.18) we get,

$$nx^{n+k-1}(1-x^{md(n-1)(n+k-1)})[x^n,y] = 0. \tag{2.19}$$

Since $a \in D(R)$, we have

$$a^{md(n-1)(n+k-1)} \in Z(R) \cap D(R) \quad \text{and} \quad Ia^{md(n-1)(n+k-1)} = 0.$$

By (2.19) we get

$$na^{n+k-1}(1-a^{md(n-1)(n+k-1)})[a^n,y] = 0.$$

This can be written as

$$(1-a^{md(n-1)(n+kl-)})na^{n+k-1}[a^n,y] = 0. \tag{2.20}$$

If $na^{n+k-1}[a^n,y] \neq 0$. Then

$$1-a^{md(n-1)(n+k-1)}\in D(R)$$

and $I(1-a^{md(n-1)(n+k-1)})=0$ and hence $I=0$, which is a contradiction. Thus we have

$$na^{n+k-1}[a^n,y]=0\,. \tag{2.21}$$

Now

$$\begin{aligned} m^2y^{m-1}y^{m-1}[a,y] &= my^{m-1}[a,y]my^{m-1} = a^k[a^n,y]my^{m-1} \\ &= a^kmy^{m-1}[a^n,y] = a^ka^{nk}[(a^n)^n,y] \\ &= a^{nk+k}n(a^n)^{n-1}[a^n,y] = a^{nk+k}na^{n-1}a^{(n-1)^2}[a^n,y] \\ &= a^{nk}a^{(n-1)^2}na^{n+k-1}[a^n,y]=0\,. \end{aligned} \tag{2.22}$$

Replacing y by $y+1$ in (2.22) and using Lemma 3, we get

$$m^2[a,y]=0 \quad \text{for all} \quad y\in R\,.$$

Replacing y by x, we get

$$m^2[x,a]=0 \quad \text{for all} \quad x\in R\,. \tag{2.23}$$

But m^2 and n^2 are relatively prime. Hence there exists integers α and β such that $m^2\alpha+n^2\beta=1$. Multiplying (2.11) by β and (2.23) by α and adding, we get

$$[x,a]=0 \quad \text{for all} \quad x\in R\,.$$

Hence $a\in Z(R)$, which proves our claim.

We know that x^{nc} and $x^{ncm}\in Z(R)$. Thus

$$\begin{aligned} (x^{nc}-x^{ncm})nx^{n+k-1}[x,y] &= nx^{nc}x^{n+k-1}[x,y]-nx^{ncm}x^{n+k-1}[x,y] \\ &= nx^{n+k-1}[x,x^{nc}y]-x^{ncm}[x,y^m] \\ &= nx^{n+k-1}[x,x^{nc}y]-[x,(x^{nc}y)^m] \\ &= nx^{n+k-1}[x,x^{nc}y]-nx^{n+k-1}[x,x^{nc}y]=0\,. \end{aligned}$$

Thus $(x-x^{ncm-nc+1})nx^{n+k-1}x^{nc-1}[x,y]=0$, i.e.

$$n(x-x^t)x^p[x,y]=0 \quad \text{for all} \quad x,y\in R \tag{2.24}$$

where $t=ncm-nc+1>1$ and $p=n+k+nc-2$.

We know that y^{md} and $y^{mdn}\in Z(R)$. Thus

$$\begin{aligned} (y^{md}-y^{mdn})my^{m-1}[x,y] &= my^{md}y^{m-1}[x,y]-my^{mdn}y^{m-1}[x,y] \\ &= my^{m-1}[xy^{md},y]-y^{mdn}x^k[x^n,y] \\ &= my^{m-1}[xy^{md},y]-x^k[(xy^{md})^n,y] \\ &= my^{m-1}[xy^{md},y]-my^{m-1}[xy^{md},y]=0 \end{aligned}$$

Thus $m(y-y^{mdn-md+1})y^{md-1}y^{m-1}[x,y]=0$. That is $m(y-y^u)y^q[x,y]=0$ for all $x,y\in R$, where $u=mdn-md+1>1$ and $q=md+m-2$. Interchanging x and y, we get

$$m(x-x^u)x^q[x,y]=0 \quad \text{for all} \quad x,y\in R\,. \tag{2.25}$$

We know that $(m,n)=1$. Hence there exists integers α and β such that $m\alpha+n\beta=1$. Multiplying (2.24) by $\beta(x-x^u)x^q$ and multiplying (2.25) by $\alpha(x-x^t)x^p$ and adding, we get

$$(x-x^t)(x-x^u)x^{p+q}[x,y]=0 \quad \text{for all} \quad x,y\in R$$

This can be written as

$$(x - x^2h(x))x^{p+q+1}[x,y] = 0 \quad \text{for all} \quad x,y \in R \tag{2.26}$$

where $h(x)$ is a polynomial in x with integers coefficients.

Suppose R is not cummutative. Then by a well known result of Herstein [6], there exists $x \in R$ such that $x - x^2h(x) \notin Z(R)$. From this it is clear that $x \notin Z(R)$. Hence x and $x - x^2h(x)$ is not a zero divisor. Hence $(x - x^2h(x))x^{p+q+1}$ is also not a zero divisor. Thus

$$[x,y] = 0 \quad \text{for all} \quad y \in R \tag{2.27}$$

This gives a contradiction. Hence R is commutative.

CASE 2: Let $n > 1$ and $m = 1$. Then (*) can be written as

$$x^k[x^n,y] = [x,y] \tag{2.28}$$

Let $e = 2^{k+n} - 2 > 0$. Then

$$\begin{aligned} e[x,y] &= 2^{k+n}[x,y] - 2[x,y] \\ &= 2^{k+n}x^k[x^n,y] - [2x,y] \\ &= (2x)^k[(2x)^n,y] - [2x,y] \\ &= [2x,y] - [2x,y] = 0 . \end{aligned}$$

All commutators are central and hence by Lemma 2,

$$[x^e,y] = ex^{e-1}[x,y] = 0 \quad \text{for all} \quad x,y \in R .$$

Hence $e^e \in Z(R)$. Now replacing x by x^n in (2.28) we get

$$x^{nk}[(x^n)^n,y] = [x^n,y] . \tag{2.29}$$

Thus

$$\begin{aligned} x^{nk}[(x^n)^n,y] &= nx^{nk}(x^n)^{n-1}[x^n,y] \\ &= nx^{nk}x^{(n-1)}x^{(n-1)^2}[x^n,y] \\ &= nx^{nk-k}x^{n+k-1}x^{(n-1)^2}[x^n,y] \\ &= nx^{n+k-1}x^{(n-1)(n+k-1)}[x^n,y] \\ &= nx^{n-1}x^{(n-1)(n+k-1)}x^k[x^n,y] \\ &= nx^{n-1}x^{(n-1)(n+k-1)}[x,y] . \end{aligned} \tag{2.30}$$

and

$$[x^n,y] = nx^{n-1}[x,y] . \tag{2.31}$$

Thus, by using (2.30) and (2.31), we can write (2.29) as

$$nx^{n-1}x^{(n-1)(n+k-1)}[x,y] = nx^{n-1}[x,y] .$$

Thus

$$nx^{n-1}(1 - x^{(n-1)(n+k-1)})[x,y] = 0 . \tag{2.32}$$

Thus, by using (2.32), we get

$$nx^{n-1}(1 - x^{e(n-1)(n+k-1)})[x,y] = 0 \tag{2.33}$$

Let $a \in D(R)$ then

$$a^{e(n-1)(n+k-1)} \in Z(R) \cap D(R) \quad \text{and} \quad Ia^{e(n-1)(n+k-1)} = 0 .$$

By using (2.33) we get

$$na^{n-1}(1-a^{e(n-1)(n+k-1)})[a,y]=0.$$

Then

$$(1-a^{e(n-1)(n+k-1)})na^{n-1}[a,y]=0. \tag{2.34}$$

If $na^{n-1}[a,y]\neq 0$. Then

$$(1-a^{e(n-1)(n+k-1)})\in D(R)$$

and $I(1-a^{e(n-1)(n+k-1)})=0$. Hence $I=0$, which is a contradiction. Thus we have

$$[a^n,y]=na^{n-1}[a,y]=0.$$

Hence $a^k[a^n,y]=[a,y]=0$ for all $y\in R$. Now $a\in Z(R)$. We know that x^e and $x^{en}\in Z(R)$. Thus

$$\begin{aligned}(x^e-x^{en+ek})[x,y] &= x^e[x,y]-x^{en+ek}[x,y]\\ &= [x^{e+1},y]-x^{en+ek}x^k[x^n,y]\\ &= [x^{e+1},y]-x^{ek}x^k[(x^{e+1})^n,y]\\ &= [x^{e+1},y]-x^{(e+1)k}[(x^{e+1})^n,y]\\ &= [x^{e+1},y]-[x^{e+1},y]=0.\end{aligned}$$

Hence $(x-x^{en+ek-e+1})x^{e-1}[x,y]=0$. If R is not commutative then by a well known result of Herstein [5] there exists $x\in R$ such that $x-x^v\notin Z(R)$ where $v=en+ek-e+1>1$. By using smaller arguments as in the last paragraph of case 1, we get a contradiction. Hence R is commutative.

We give examples which show that all the hypotheses of our main theorem are essential. The following example show that R is not cummutative if m and n are not relatively prime or the ring is without unity in the hypothesis of our main theorem.

EXAMPLE 1. Let

$$R=\left\{\begin{pmatrix}0 & a & b\\ 0 & 0 & c\\ 0 & 0 & 0\end{pmatrix} : a,b,c\in F,\ F:\ \text{field}\right\}$$

Then R is a ring without unity satisfying $x^k[x^2,y]=[x,y^3]$ and for all non-negative integer k. But R is not commutative.

EXAMPLE 2. Let

$$R=\left\{\begin{pmatrix}a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix} : a,b,c,d\in GF(2)\right\}$$

Then R is a ring with unity satisfying $x^k[x^4,y]=[x,y^4]$ for all $x,y\in R$ and for all non-negative integer k. But R is not commutative.

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