Internat. J. of Math. & Math. Sci.
VOL. 17 NO. 2 (1994) 209-216

ON A SUBCLASS OF C^1 FUNCTIONS FOR WHICH THE LAGRANGE INTERPOLATION YIELDS THE JACKSON ORDER OF APPROXIMATION

XIN LI

Department of Mathematics,
University of Central Florida,
Orlando, FL 32816

(Received November 4, 1992 and in revised form January 8, 1993)

ABSTRACT. We continue the investigation initiated by Mastroianni and Szabados on question whether Jackson's order of approximation can be attained by Lagrange interpolation for a wide class of functions. Improving a recent result of Mastroianni and Szabados, we show that for a subclass of C^1 functions the local order of approximation given by Lagrange interpolation can be much better (of at least $O(\frac{1}{n})$) than Jackson's order.

KEY WORDS AND PHRASES. Interpolation, Order of approximation.
1991 AMS SUBJECT CLASSIFICATION CODES. 41A05, 41A25.

1. INTRODUCTION.

Let X be an infinite triangular matrix of points with n-th row entry $(x_{nn}, x_{n-1,n}, \dots, x_{2,n}, x_{1,n})$ satisfying

$$-1 \le x_{nn} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n} \le 1, \quad n = 1, 2, \dots$$

(we write x_k for $x_{k,n}$ when there is no confusion), and let

$$w_n(x) = w_n(x, X) := \prod_{k=1}^{n} (x - x_{k,n}), \quad n = 1, 2, \dots.$$

For a function f defined on points contained in X, define

$$L_n(f, X, x) := \sum_{k=1}^{n} \frac{f(x_k) w_n(x)}{w_n'(x_k)(x - x_k)},$$

then $L_n(f, X, x)$ is the unique Lagrange interpolation polynomial of f of degree $n-1$.

It is well-known by Faber's classical theorem that for any matrix X, there is a function $f \in C[-1,1]$ such that $L_n(f, X, x)$ does not converge to f on $[-1,1]$ as $n \to \infty$. On the other hand, by Jackson's theorem (and its pointwise generalization given by Telyakowski and Gopengauz), for any function f with $f^{(r)} \in C[-1,1]$, there is a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ with $p_n \in \mathcal{P}_{n-1}$ (the set of polynomials of degree at most $n-1$) such that

$$|f(x) - p_n(x)| = O[\left(\frac{\sqrt{1-x^2}}{n}\right)^r \omega(f^{(r)}, \frac{\sqrt{1-x^2}}{n})], \quad (n \to \infty). \tag{1}$$

In [2], Mastroianni and Szabados initiated the investigation of the following problem.

PROBLEM. Find a reasonably wide class of functions and some matrices of interpolation X such that the error of the corresponding Lagrange interpolation yields Jackson or possibly Telyakowski-Gopengauz order of convergence.

It is proved in [2] that if we use the matrix U formed by the extremal points of the Chebyshev polynomials, i.e., $x_{kn} = \cos t_k$, $t_k = t_{kn} := (k-1)\pi/(n-1)$, $k = 1, 2, ..., n$, then

THEOREM 1. ([2, Theorem 1]) If $f \in C[-1,1]$ and there is a partition of $[-1,1] : -1 = a_s < a_{s-1} < \cdots < a_0 = 1$ such that every $f|_{[a_{j+1}, a_j]}$ is a polynomial ($j = 0, 1, ..., s-1$), then, for $|x| \le 1$,

$$|f(x) - L_n(f,U,x)| = O(\frac{\sqrt{1-x^2}}{n}) \min(1, \frac{1}{n \min_{1 \le j \le s-1} |x - a_j|}), \quad (n \to \infty). \tag{2}$$

As noted in [2], function f satisfying the condition stated in Theorem 1 is in Lip1. For such function f, the order given by (1) is $O(\frac{\sqrt{1-x^2}}{n})$. Therefore, (2) can give the same order as in (1) so yields an answer to the above problem. Further, the estimation in (2) shows that the order of approximation by Lagrange interpolation can be improved (to $O(\frac{1}{n^2})$) locally if we stay away from the singularities a_j's.

Mastroianni and Szabados also show how the above phenomenon carries over to smoother functions by proving the following result.

THEOREM 2. ([2, Theorem 2]) In addition to the assumption in Theorem 1, we assume $f' \in C[-1,1]$. Then, for $|x| \le 1$,

$$|f(x) - L_n(f,U,x)| = O(\frac{\sqrt{1-x^2}}{n^2}), \quad (n \to \infty). \tag{3}$$

Since function f in Theorem 2 satisfies $f' \in$ Lip1, the order given by (1) for such f is $O(\frac{1-x^2}{n^2})$. So (3) does not macth (1) in the order of approximation. It is then natural to ask: Under the same assumption as in Theorem 2, is it possible to get order $O(\frac{1-x^2}{n^2})$ instead of $O(\frac{\sqrt{1-x^2}}{n^2})$ in (3) ? We ask even further: For functions in Theorem 2, can the order of approximation by Lagrange interpolation be improved locally as in Theorem 1 ?

The goal of this note is to answer these questions. We will improve the above results of Mastroianni and Szabados. Preciesly, we will show that under the assumption in Theorem 2 we *can* get order $O(\frac{1-x^2}{n^2})$ as well as better local estimation (as in Theorem 1 above). The paper is organized as follows. In section 2, we state our main results. Then in sections 3-5, we give the proofs.

2. MAIN RESULTS.

As suggested in [2], function $|x|$ is a representative of the functions considered in Theorem 1. The following result tells us that the local order of approximation $O(\frac{1}{n^2})$ in Theorem 1 is sharp in general (when we stay away from the singularities).

PROPOSITION 3. We have

$$|\, |x| - L_n(|t|, U, x) \,| = \frac{|w_n(x)|(\frac{n}{2} - [\frac{n}{2}])}{n^2|x|} + O(\frac{1}{n^3|x|^2}), \quad (n \to \infty),$$

where "O" is locally uniform for $x \in [-1,1] \backslash \{0\}$.

The proof of Proposition 3 is given in section 4.

We will consider classes of functions more general than those in Theorems 1 and 2. We call a function *piecewisely global analytic on* $[-1,1]$ *with singularities* $\{a_j\}_{j=1}^{s-1}$ if

$$-1 =: a_s < a_{s-1} < \cdots < a_1 < a_0 := 1$$

and every $f|_{[a_{j+1}, a_j]}$ has an analytic continuation to $[-1,1]$, ($j = 0, 1, ..., s-1$).

THEOREM 4. Let f be piecewisely global analytic on $[-1,1]$ with singularities $\{a_j\}_{j=1}^{s-1}$. Then
(i) if $f \in C[-1,1]$, we have (2);
(ii) if $f' \in C[-1,1]$, we have

$$|f(x) - L_n(f,U,x)| = O(1) \min\{\frac{1-x^2}{n^2}, \frac{\sqrt{1-x^2}}{n^3 \min_{1 \le j \le s-1} |x - a_j|}\}, \tag{4}$$

as $n \to \infty$, where O is uniform in $x \in [-1,1]$.

The proof of Theorem 4 is given in section 5.

We will use the following notation for "trapesoidal" sum:

$$\sum_{k=K_1}^{K_2}{}^{*}\ h_k := \frac{1}{2}h_{K_1} + h_{K_1+1} + \cdots + h_{K_2-1} + \frac{1}{2}h_{K_2}. \tag{5}$$

The following elementary result plays a crucial role in our proofs.

LEMMA 5. (i) If f'' is monotone on $[1, 2m+1]$, then

$$\sum_{k=1}^{2m+1}{}^{*}\ (-1)^{k+1}f(k) = \frac{1}{4}[f'(2m+1) - f'(1)] + O(|f''(2m+1) - f''(1)|), \quad (m \to \infty).$$

(ii) If $|f'''|$ is integrable on $[1, 2m+1]$, then

$$\sum_{k=1}^{2m+1}{}^{*}\ (-1)^{k+1}f(k) = \frac{1}{4}[f'(2m+1) - f'(1)] + O\Big(\int_1^{2m+1} |f'''(t)|dt\Big), \quad (m \to \infty).$$

It is easy to check Lemma 5. For reader's convenience, a proof of it is provided in next section.

3. PROOF OF LEMMA 5.

For $k = 1, 2, \ldots, m$, using the mean value theorem for differrences of order 1 and 2, we can write

$$\begin{aligned} &\frac{1}{2}f(2k-1) - f(2k) + \frac{1}{2}f(2k+1) - \frac{1}{4}[f'(2k+1) - f'(2k-1)] \\ = &\ \frac{1}{2}f''(\xi_k) - \frac{1}{4}f''(\varsigma_k)2 = \frac{1}{2}[f''(\xi_k) - f''(\varsigma_k)], \end{aligned} \tag{6}$$

where $\xi_k,\ \varsigma_k \in (2k-1, 2k+1)$.

(i) If f'' is monotone, then from (6)

$$\begin{aligned} &|\frac{1}{2}f(2k-1) - f(2k) + \frac{1}{2}f(2k+1) - \frac{1}{4}[f'(2k+1) - f'(2k-1)]| \\ \le &\ \frac{1}{2}|f''(2k+1) - f''(2k-1)|. \end{aligned} \tag{7}$$

Thus

$$\begin{aligned} &|\sum_{k=1}^{2m+1}{}^{*}\ (-1)^{k+1}f(k) - \frac{1}{4}[f'(2m+1) - f'(1)]| \\ = &\ |\sum_{k=1}^{m}\{\frac{1}{2}f(2k-1) - f(2k) + \frac{1}{2}f(2k+1) - \frac{1}{4}[f'(2k+1) - f'(2k-1)]\}| \\ \le &\ \sum_{k=1}^{m}|\frac{1}{2}f(2k-1) - f(2k) + \frac{1}{2}f(2k+1) - \frac{1}{4}[f'(2k+1) - f'(2k-1)]| \\ \le &\ \frac{1}{2}|f''(2m+1) - f''(1)|. \end{aligned} \tag{8}$$

Here in the last inequality we used (7) and monotonicity of f''. This completes the proof of (i) of the lemma.

(ii) If $|f'''|$ is integrable, then from (6)

$$\frac{1}{2}f(2k-1) - f(2k) + \frac{1}{2}f(2k+1) - \frac{1}{4}[f'(2k+1) - f'(2k-1)] = \frac{1}{2}\int_{\varsigma_k}^{\xi_k} f'''(t)dt,$$

with $\xi_k,\ \varsigma_k \in (2k-1, 2k+1)$, so

$$|\frac{1}{2}f(2k-1) - f(2k) + \frac{1}{2}f(2k+1) - \frac{1}{4}[f'(2k+1) - f'(2k-1)]| \le \frac{1}{2}\int_{2k-1}^{2k+1} |f'''(t)|dt.$$

Hence as in (8), we can get

$$|\sum_{k=1}^{2m+1}{}^{*}\ (-1)^{k+1}f(k) - \frac{1}{4}[f'(2m+1) - f'(1)]| \le \frac{1}{2}\int_1^{2m+1} |f'''(t)|dt.$$

This completes our proof of Lemma 5.

4. PROOF OF PROPOSITION 3.

It is known that [3, formula (1.98), p. 38]

$$L_n(f,U,x) = \sum_{k=1}^{n}{}^{*}\frac{(-1)^k f(x_k) w_n(x)}{(n-1)(x-x_k)}, \quad \text{for any } f, \tag{9}$$

where $\sum^*$ has the same meaning as indicated in (5).

Assume $x < 0$; the proof for the case when $x > 0$ is similar. We have

$$\begin{aligned} R_n(x) &:= \big|\, |x| - L_n(|t|,U,x)\,\big| = |-x - L_n(|t|,U,x)| \\ &= |-L_n(t,U,x) - L_n(|t|,U,x)| = |L_n(t+|t|,U,x)| \\ &= \frac{|w_n(x)|}{n-1}\left|\sum_{k=1}^{n}{}^{*}\frac{(-1)^k(x_k+|x_k|)}{x-x_k}\right| \\ &= \frac{2|w_n(x)|}{n-1}\left|\sum_{k=1}^{[\frac{n}{2}]}\frac{(-1)^k x_k}{(1+\delta_{k1})(x_k-x)}\right|. \end{aligned}$$

Define $\Phi(y) := \cos y/(\cos y - x)$ and $\phi(t) := \Phi(\pi(t-1)/(n-1))$, then

$$R_n(x) = \frac{2|w_n(x)|}{n-1}\left|\sum_{k=1}^{[\frac{n}{2}]}\frac{(-1)^{k+1}\phi(k)}{1+\delta_{k1}}\right|$$

Denote $n' := [\frac{n}{2}]$. We now estimate the summation

$$S_n := \sum_{k=1}^{n'}\frac{(-1)^{k+1}\phi(k)}{1+\delta_{k1}} = \frac{1}{2}\phi(1) - \phi(2) + \cdots + (-1)^{n'+1}\phi(n').$$

According to whether n' is *even* or *odd*, we can write

$$S_n = \{\frac{1}{2}\phi(1) - \phi(2) + \cdots + \frac{1}{2}\phi(n'-1)\} + \frac{1}{2}\phi(n'-1) - \phi(n')$$

or

$$S_n = \{\frac{1}{2}\phi(1) - \phi(2) + \cdots + \frac{1}{2}\phi(n')\} + \frac{1}{2}\phi(n'),$$

respectively. Let us first consider the case when n' is even. Note that

$$\Phi(y) = -\frac{x\sin y[(x^2-5) + 4x\cos y - \sin^2 y]}{(\cos y - x)^4} < 0, \quad \text{for } \ y \in (0,\frac{\pi}{2}),$$

so ϕ'' is decreasing on $[1, n'-1]$. Thus, by Lemma 5 (i),

$$\begin{aligned} S_n &= \frac{1}{4}[\phi'(n'-1) - \phi'(1)] + O(|\phi''(n'-1) - \phi''(1)|) \\ &\quad + \frac{1}{2}\phi(n'-1) - \phi(n'). \end{aligned} \tag{10}$$

Now we pause our proof to state the following lemma which can be checked by straightforward computation and hence the proof is omitted.

LEMMA 6. Denote $d := \frac{n}{2} - n'$. We have (i)

$$\phi(n'-1) = \frac{\pi(-\frac{3}{2}-d)}{nx} + O(\frac{1}{n^2|x|^2}), \quad (n\to\infty),$$

and

$$\phi(n') = \frac{\pi(-\frac{1}{2}-d)}{nx} + O(\frac{1}{n^2|x|^2}), \quad (n\to\infty).$$

(ii) $\phi'(1) = 0$ and

$$\phi'(n'-1) = \frac{\pi}{nx} + O(\frac{1}{n^2|x|^2}), \quad (n\to\infty).$$

(iii)

$$|\phi''(t)| = O(\frac{1}{n^2|x|^2}), \quad (n \to \infty).$$

Here all the O's are locally uniform for $x \in [-1,1]\backslash\{0\}$.

We now continue our proof. With Lemma 6, (10) gives us

$$S_n = \frac{\pi d}{2nx} + O(\frac{1}{n^2|x|^2}), \quad (n \to \infty). \tag{11}$$

Similarly, we can show that (11) is true when n' is odd. This completes our proof.

5. PROOF OF THEOREM 4.

We prove only (ii); the proof of (i) is simpler and requires minor modifications of that given in [2]. These modifications are also contained in the following proof of (ii). Some ideas of our proof are borrowed from [2].

Let $p_i(z)$ bs analytic on $[-1,1]$ and $f|_{[a_{i+1},a_i]}(z) = p_i(z)$ on $[a_{i+1}, a_i] =: I_i$, $i = 0,1,...,s-1$. For $x \in [-1,1]$ (fixed), there is a $j(x) =: j$ such that $x \in I_j$. Then

$$\begin{aligned} R_n(x) &:= |f(x) - L_n(f,U,x)| = |p_j(x) - L_n(f,U,x)| \\ &= |p_j(x) - L_n(p_j,U,x) + L_n(p_j - f,U,x)|. \end{aligned}$$

It is known that (cf., e.g., [1,Theorem 2, p. 66]) for a function g analytic on $[-1,1]$,

$$|g(x) - L_n(g,U,x)| = O(\frac{|w_n(x)|}{\rho_g^n}), \quad \text{for some } \rho_g > 1.$$

So for some $\rho := \rho_{p_i} > 1$,

$$R_n(x) = O(\frac{|w_n(x)|}{\rho^n}) + |L_n(p_j - f,U,x)|.$$

It has already been shown (with minor modification for p_i being analytic instead of a polynomial function) in the proof of Theorem 2 in [2] that

$$L_n(p_j - f,U,x) = O(\frac{|w_n(x)|}{n^2}). \tag{12}$$

We now prove

$$L_n(p_j - f,U,x) = O(\frac{|w_n(x)|}{n^3 \min_{1\le j\le s-1}|x-a_j|}). \tag{13}$$

Define

$$\Phi(y) := \frac{p_j(\cos y) - f(\cos y)}{x - \cos y}$$

and $\phi(t) := \Phi((t-1)\pi/(n-1))$. Then, using (9),

$$|L_n(p_j - f,U,x)| = \frac{|w_n(x)|}{n-1}\left|\sum_{\{k:\ x_k\notin I_j\}} \frac{(-1)^k\phi(k)}{1+\delta_{k1}+\delta_{kn}}\right| \tag{14}$$

$$= \frac{|w_n(x)|}{n-1}\left|(\sum_{x_k<a_{j+1}} + \sum_{x_k>a_j})\frac{(-1)^k\phi(k)}{1+\delta_{k1}+\delta_{kn}}\right|.$$

The two sums can be estimated similarly. Let us consider $\sum_{x_k>a_j}$. Let μ satisfy $x_{\mu+1} \le a_j < x_\mu$, then, as in [2],

$$\sum_{x_k>a_j} \frac{(-1)^{k+1}\phi(k)}{1+\delta_{k1}} = \sum_{k=1}^{2[\mu/2]-1}{}^{*}(-1)^{k+1}\phi(k) + \frac{1+(-1)^{\mu+1}}{4}\phi(\mu-2) \tag{15}$$

$$+\frac{3(-1)^\mu - 1}{4}\phi(\mu-1) + (-1)^{\mu+1}\phi(\mu).$$

For n large enough, $x_{\mu-2},\ x_{\mu-1},\ x_\mu \in I_{j-1}$ and

$$\phi(m) = \frac{(p_j - p_{j-1})(x_m) - (p_j - p_{j-1})(a_j) - (p'_j - p'_{j-1})(a_j)(x_m - a_j)}{(a_j - x) + (x_m - a_j)} \tag{16}$$

by the fact that $f' \in C[-1,1]$ (so $p_j(a_j) = p_{j-1}(a_j)$, and $p'_j(a_j) = p'_{j-1}(a_j)$), where $m = \mu - 2$, $\mu - 1$, μ. Then the Taylor theorem implies

$$\phi(m) = \frac{\frac{1}{2}[p''_j(\xi) - p''_{j-1}(\xi)](x_m - a_j)^2}{(a_j - x) + (x_m - a_j)}, \tag{17}$$

so

$$|\phi(m)| \leq \frac{\max_{z\in[-1,1]} |p''_j(z) - p''_{j-1}(z)|(x_m - a_j)^2}{2(a_j - x)} = O(\frac{(x_m - a_j)^2}{|x - a_j|}).$$

But $(x_m - a_j)^2 \leq (x_{\mu+1} - x_{\mu-2})^2 = O(\frac{1}{n^2})$, hence

$$|\phi(m)| = O(\frac{1}{n^2|x - a_j|}), \quad (n \to \infty), \tag{18}$$

for $m = \mu - 2, \mu - 1, \mu$. Next, we estimate $\sum^*$ in (15). Note that when n is large enough, $x_{2k-1}, x_{2k}, x_{2k+1}$ will be contained in at most two I_i's. So, for large n, we can write

$$\sum_{k=1}^{2[\mu/2]-1} {}^*(-1)^{k+1}\phi(k) \tag{19}$$
$$= \sum_{i=0}^{j-1}\Big(\sum_{x_{2k-1},x_{2k+1}\in I_i} + \sum_{x_{2k+1}<a_i<x_{2k-1}} \Big)[\frac{1}{2}\phi(2k-1) - \phi(2k) + \frac{1}{2}\phi(2k+1)].$$

Step I: Estimation of $\sum_{x_{2k+1}<a_i<x_{2k-1}}$ (when it is not an empty sum).

The sum contains only one term, i.e.,

$$\sum_{x_{2k+1}<a_i<x_{2k-1}} [\frac{1}{2}\phi(2k-1) - \phi(2k) + \frac{1}{2}\phi(2k+1)] = \frac{1}{2}\phi(2k-1) - \phi(2k) + \frac{1}{2}\phi(2k+1).$$

Assume $a_i \leq x_{2k}$. (The case when $x_{2k} < a_i$ can be handled similarly.) Then

$$\begin{aligned} & \frac{1}{2}\phi(2k-1) - \phi(2k) + \frac{1}{2}\phi(2k+1) \qquad (20)\\ = & \frac{p_j(x_{2k-1}) - p_{i-1}(x_{2k-1})}{2(x_{2k-1} - x)} - \frac{p_j(x_{2k}) - p_{i-1}(x_{2k})}{x_{2k} - x} + \frac{p_j(x_{2k+1}) - p_i(x_{2k+1})}{2(x_{2k+1} - x)} \\ = & \left[\frac{p_j(x_{2k-1})}{2(x_{2k-1} - x)} - \frac{p_j(x_{2k})}{x_{2k} - x} + \frac{p_j(x_{2k+1})}{2(x_{2k+1} - x)}\right] \\ & - \left[\frac{p_{i-1}(x_{2k-1})}{2(x_{2k-1} - x)} - \frac{p_{i-1}(x_{2k})}{x_{2k} - x} + \frac{p_{i-1}(x_{2k+1})}{2(x_{2k+1} - x)}\right] + \frac{p_{i-1}(x_{2k+1}) - p_i(x_{2k+1})}{2(x_{2k+1} - x)} \\ =: & \ T_1 - T_2 + T_3. \end{aligned}$$

To estimate T_1, we use the mean value theorem to get

$$|T_1| = O(\frac{1}{n^2}) \max_{x_{2k+1}\leq y\leq x_{2k-1}} \left|\frac{d^2}{dy^2}(\frac{p_j(y)}{y - x})\right|. \tag{21}$$

We may assume $|x_{2k+1} - a_i| < (a_i - a_{i+1})/2$ by assuming n is large enough. So, for $x_{2k+1} \leq y \leq x_{2k-1}$,

$$|y - x| \geq |x_{2k+1} - a_{i+1}| \geq \frac{a_i - a_{i+1}}{2},$$

where the first inequality depends on the fact that $x \leq a_j \leq a_{i+1}$. Thus (21) implies $T_1 = O(\frac{1}{n^2})$. Similarly, we can show $T_2 = O(\frac{1}{n^2})$. On the other hand, proceeding as in (16) and (17), we can get

$$T_3 = O(\frac{(x_{2k+1} - a_i)^2}{|x_{2k+1} - x|}).$$

Note that $(x_{2k+1}-a_i)^2 \le (x_{2k+1}-x_{2k})^2 = O(\frac{1}{n^2})$ and $|x_{2k+1}-x| \ge (a_i - a_{i+1})/2$ (for n large enough), it then follows that $T_3 = O(\frac{1}{n^2})$.

Using the above estimation on T_1, T_2 and T_3 in (20), we obtain

$$\sum_{x_{2k+1}<a_i<x_{2k-1}} [\frac{1}{2}\phi(2k-1) - \phi(2k) + \frac{1}{2}\phi(2k+1)] = O(\frac{1}{n^2}).$$

Step II: Estimation of $\sum_{i=0}^{j-1}\sum_{x_{2k+1},x_{2k-1}\in I_i}$.

Let l_i and m_i be odd integers satisfying

$$x_{m_i+2} < a_{i+1} \le x_{m_i}, \quad x_{l_{i+1}} \le a_{i+1} < x_{l_{i+1}-2}, \qquad \text{for } i = 0,1,...,j-2,$$

$l_0 := 1$ and $m_{j-1} := 2[\mu/2]-1$. Then $m_i > l_i \ge m_{i-1}$ and

$$\sum_{x_{2k+1},x_{2k-1}\in I_i} [\frac{1}{2}\phi(2k-1) - \phi(2k) + \frac{1}{2}\phi(2k+1)] = \sum_{k=l_i}^{m_i}{}^*(-1)^{k+l_i}\phi(k).$$

Applying Lemma 5 (ii), we have

$$\sum_{k=l_i}^{m_i}{}^*(-1)^{k+l_i}\phi(k) = \frac{1}{4}[\phi'(m_i) - \phi'(l_i)] + O(\int_{l_i}^{m_i} |\phi'''(t)|dt).$$

So

$$\begin{aligned}&\sum_{i=0}^{j-1}\sum_{x_{2k+1},x_{2k-1}\in I_i} [\frac{1}{2}\phi(2k-1) - \phi(2k) + \frac{1}{2}\phi(2k+1)] \qquad (22)\\ =\ & \frac{1}{4}[\phi'(2[\frac{\mu}{2}]-1) - \phi'(1)] + \frac{1}{4}\sum_{i=0}^{j-2}[\phi'(m_i) - \phi'(l_{i+1})] + O(\int_1^{2[\mu/2]-1} |\phi'''(t)|dt).\end{aligned}$$

We first note that $\phi'(1) = 0$ and with $x^* := \cos\frac{2[\mu/2]-2}{n-1}\pi$

$$\begin{aligned}\phi'(2[\frac{\mu}{2}]-1) &= \frac{\pi}{n-1}\left[\frac{p_j(x^*) - p_{j-1}(x^*)}{(x^*-x)^2} - \frac{p_j'(x^*) - p_{j-1}'(x^*)}{(x^*-x)}\right]\sin\frac{2[\frac{\mu}{2}]-2}{n-1}\pi \\ &= O(\frac{1}{n})\left\{\frac{\frac{1}{2}[p_j''(\xi) - p_{j-1}''(\xi)](x^*-a_j)^2}{(x^*-x)^2} - \frac{[p_j''(\varsigma) - p_{j-1}''(\varsigma)](x^*-a_j)}{(x^*-x)}\right\},\end{aligned}$$

where $\xi, \varsigma \in (a_j, x^*)$. Then, as $n \to \infty$,

$$|\phi'(2[\frac{\mu}{2}]-1)| = O(\frac{1}{n})\frac{|x^*-a_j|}{|x^*-x|} = O(\frac{1}{n})\frac{|x^*-x_{\mu+1}|}{|a_j-x|} = O(\frac{1}{n^2|a_j-x|}), \qquad (23)$$

since $x^* - x_{\mu+1} = O(\frac{1}{n})$. Next, note that for $i = 0,1,...,j-2$, $|m_i - l_{i+1}| = O(1)$. Together with $\max_{1\le t\le l_{j-1}}|\phi''(t)| = O(\frac{1}{n^2})$, we get

$$\sum_{i=0}^{j-2}[\phi'(m_i) - \phi'(l_{i+1})] = O(\frac{1}{n^2}), \qquad (n \to \infty). \qquad (24)$$

Finally, using the fact that (from $f' \in C[-1,1]$)

$$p_j(\cos\frac{t-1}{n-1}\pi) - p_{j-1}(\cos\frac{t-1}{n-1}\pi) = O(|\cos\frac{t-1}{n-1}\pi - a_j|^2)$$

and

$$p_j'(\cos\frac{t-1}{n-1}\pi) - p_{j-1}'(\cos\frac{t-1}{n-1}\pi) = O(|\cos\frac{t-1}{n-1}\pi - a_j|),$$

together with $|\cos\frac{t-1}{n-1}\pi - x| \ge |\cos\frac{t-1}{n-1}\pi - a_j|$ for $t \in [1,\mu]$, it is straightforward to get

$$\phi'''(t) = O(\frac{1}{n^3(\cos\frac{t-1}{n-1}\pi - x)^2}), \qquad t \in [1,\mu].$$

Note that (cf. [2]) there exists a $c_1 > 0$ such that $c_1 n \le \mu \le n$, so it is possible to find a number $c > 0$ such that $\cos\frac{t-1}{n-1}\pi - x \ge \cos\frac{t-1}{n-1}\pi - x_\mu + a_j - x \ge c\frac{\mu-t}{n} + (a_j - x)$. Then

$$\int_1^{2[\mu/2]-1} \frac{1}{(\cos\frac{t-1}{n-1}\pi - x)^2} dt \le \int_1^{2[\mu/2]-1} \frac{1}{(c\frac{\mu-t}{n} + a_j - x)^2} dt \le \frac{n}{c|x-a_j|}.$$

Therefore,

$$\int_1^{2[\mu/2]-1} |\phi'''(t)| dt = O(\frac{1}{n^2|x-a_j|}), \quad (n \to \infty). \tag{25}$$

Using (23-25) in (22) we obtain

$$\sum_{i=0}^{j-1} \sum_{x_{2k+1}, x_{2k-1} \in I_i} [\frac{1}{2}\phi(2k-1) - \phi(2k) + \frac{1}{2}\phi(2k+1)] = O(\frac{1}{n^2|x-a_j|}), \ (n \to \infty).$$

Combining estimations obtained in Steps I and II and using (19) we have

$$\sum_{k=1}^{2[\mu/2]-1} {}^*(-1)^{k+1}\phi(k) = O(\frac{1}{n^2|x-a_j|}), \ (n \to \infty). \tag{26}$$

Obviously, (13) follows from (14) and (26).

Finally, we establish (4) by using (12) and (13).

Since $w_n(x) = \sin\theta\sin(n-1)\theta$ (with $x = \cos\theta$), $|w_n(x)| \le \sqrt{1-x^2}$. So (12) and (13) immediately gives us

$$|f(x) - L_n(f,U,x)| = O(\frac{\sqrt{1-x^2}}{n^2}) \min(1, \frac{1}{n \min_{1\le j\le s-1}|x-a_j|}), \quad (n \to \infty). \tag{27}$$

It remains to verify

$$|f(x) - L_n(f,U,x)| = O(\frac{1-x^2}{n^2}), \qquad (n \to \infty). \tag{28}$$

Let $\delta := 2^{-1}\min(a_{s-1}+1, 1-a_1)$. If $x \in [-1+\delta, 1-\delta]$, then $\sqrt{1-x^2} = O(1-x^2)$, so (27) yields (28). If $x \notin [-1+\delta, 1-\delta]$, then $|x - a_i| \ge \delta$, $i = 1, 2, ..., s-1$. So (13) gives us

$$|f(x) - L_n(f,U,x)| = O(\frac{|\sin\theta\sin(n-1)\theta|}{n^3}), \quad (n \to \infty),$$

for $x = \cos\theta \notin [-1+\delta, 1-\delta]$. Using inequality $|(\sin(n-1)\theta)/\sin\theta| \le (n-1)$, it then follows

$$|f(x) - L_n(f,U,x)| = O(\frac{\sin^2\theta}{n^2}) = O(\frac{1-x^2}{n^2}), \ (n \to \infty)$$

for $x \notin [-1+\delta, 1-\delta]$. So (28) holds for all $x \in [-1,1]$. This completes our proof.

REFERENCES

1. Gaier, D. Lectures on Complex Approximation, Birkhauser, Boston, 1987.

2. Mastroianni, G. and Szabados, J. On some classes of functions where the Jackson order of approximation is attained by Lagrange interpolation, manuscript, 1992.

3. Rivlin, T. J. Chebyshev Polynomials (2nd Edition), John Wiley & Sons, New York, 1990.