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SOME RESULTS ON INVARIANT APPROXIMATION

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ABSTRACT. In this paper results on Invariant approximations, extending and unifying earlier results are given. Several interesting results are derived as corollaries.

KEY WORDS AND PHRASES. Nonexpansive map, set of best approximants, starshaped set, demiclosed, approximatively compact set.

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1. INTRODUCTION

Several mathematicians have given results dealing with invariant approximation. The earlier results due to Meinardus [7] and Brosowski [1] have been the main source of inspiration for researchers in the field.

The following is due to Brosowski.

Let f be a contractive linear operator of a normed space X. Let $x_0 \in X$ be an invariant point and C an invariant set under f. If the set of best C-approximants to x_0 is nonempty compact and convex then it contains an f-invariant point.

Recently Carbone [2], [3], Habiniak [5], Hicks and Humphries [6], Narang [8], Rao and Mariadoss [9] Sahab, Khan and Sessa [10], K.L.Singh [11], Singh [12], [13] and Subrahmanyam [15] gave extensions of the above theorem, and proved some further related results.

The purpose of this paper is to extend and unify earlier results.

We need the following preliminaries.

If $f : X \to X$ where X is a normed linear space, and $\|fx - fy\| \leq \|x - y\|$ for $x, y \in X$, then f is called a *nonexpansive* map.

We denote by $F(f)$, the set of fixed points of f. If $x_0 = fx_0$, and $\|fx - fx_0\| \leq \|x - x_0\|$ for all $x \in X$ then f is said to be *quasi-nonexpansive*. A quasi-nonexpansive map need not be continuous.

Let C be a closed, convex subset of a normed linear space X. The set of best C-approximants to x consists of all $y \in C$ such that

$$\|y - x\| = d(x, C) = inf\{\|x - z\| : z \in C\}.$$

A set C is said to be *star-shaped* if there exists at least one point $p \in C$ such that $\lambda x + (1 - \lambda)p \in C$ for all $x \in C$ and $0 < \lambda < 1$. In this case p is called the *star-center* of C.

f is said to be *demiclosed* if $x_n \rightharpoonup x$ weakly and $fx_n \to y$ strongly imply that $y = fx$.

The following result will be used in the proof of our theorem (see Subrahmanyam [15], Zhang [19]).

Let X be a Banach space and C a nonempty weakly compact starshaped subset of X. If $f : C \to C$ is nonexpansive and $I - f$ is demiclosed then f has a fixed point

Note: The above theorem has been also proved when $f : C \to X$ is nonexpansive and either $f(\partial C) \subseteq C$ or f is an inward map.

Remark

1. In case C is a compact, convex subset of X and $f : C \to X$ a nonexpansive map with $f(\partial C) \subseteq C$, then f has a fixed point. This is valid even when C is star-shaped but not necessarily convex.
2. If C is weakly compact convex and $I - f$ is demiclosed then a nonexpansive map $f : C \to X$ with $f(\partial C) \subseteq C$ has a fixed point.

We can replace convexity by star-shaped condition without changing the conclusion.

If X is a normed linear space and M is a subspace of X, and $x_0 \in X$, then $d(x_0, M) = inf\{\|x_0 - y\| : y \in M\}$ denotes the distance of x_0 to the set M. $P_M(x_0) = \{y \in M : \|x_0 - y\| = d(x_0, M)\}$ is the set of best approximations of x_0 in M. It is well known that the set $P_M(x)$ is closed and convex for any subspace M of X (for details see [4]).

2. MAIN RESULTS

Now we state our result.

THEOREM 1 — *Let X be a Banach space. Suppose $f : X \to X$ is such that $f(\partial C) \subseteq C$ and $fx_0 = x_0$ for $x_0 \in X$. If D, the set of best C-approximants to x_0, is weakly compact star-shaped and*

i) f is nonexpansive on D;

ii) $\|fx - fx_0\| \leq \|x - x_0\|$ for all $x \in D$;

iii) $I - f$ is demiclosed on D

then f has a fixed point closest to x_0.

PROOF First we show that $f : D \to D$, i.e. f is a self-map on D.

Let $y \in D$. Then

$$\|fy - x_0\| = \|fy - fx_0\| \leq \|y - x_0\|.$$

This implies that fy is also closest to x_0 so $fy \in D$.

Let p be the star center of D.

Define $f_{k_n}x = k_n fx + (1 - k_n)p$, where $\{k_n\}$ is a sequence of reals, $0 < k_n < 1$ converging to 1 as $n \to \infty$ for $x \in D$.

Each f_{k_n} is contraction on D with contraction constant $k_n < 1$, so $f_{k_n}x_{k_n} = x_{k_n}$ for $n = 1, 2, \ldots..$ (see Subrahmanyam [15] for details).

Since D is weakly compact therefore $\{x_{k_n}\}$ has a convergent subsequence converging weakly to $\bar{x}$ say. For the sake of convenience let $x_{k_i} \to \bar{x}$.

Now,

$\|x_{k_i} - fx_{k_i}\| = \|f_{k_i}x_{k_i} - fx_{k_i}\| = (k_i - 1)\|fx_{k_i}\| + (1 - k_i)p \to 0$ as $i \to \infty$ ($\{fx_{k_i}\}$ is bounded).

Since $I - f$ is demiclosed so $0 = (I - f)\bar{x}$, i.e., $\bar{x} = f\bar{x}$. Thus f has a fixed point x closest to x_0.

Note

If we take D as a compact convex set and $f : D \to D$ continuous then we derive the same conclusion using Schauder fixed point theorem. In this case, we have to assume that $f: D \to D$. In our theorem f is nonexpansive so $f: D \to D$ follows.

We derive the following results as corollaries.

Corollary 1. If C is compact, convex then $I - f$ demiclosed is not needed and we get the result (see [19]).

Corollay 2. If C is weakly compact, convex and other conditions are the same then we get the result.

Recall that every convex set is starshaped. However, the converse is not true.

In this section we discuss a result of Rao ad Mariadoss [9] given for approximatively compact sets.

A set $C \subseteq X$ is called *approximatively compact* if for each $x \in X$ and every sequence $\{x_n\}$ in C with $\lim_{n \to \infty} \|x - x_n\| = d(x, C)$, there exists a subsequence $\{x_{n_i}\}$ converging to an element of C.

A compact set is always approximatively compact but not conversely. A closed unit ball in infinite dimensional Hilbert space is approximatively compact but not compact.

Let C be a nonempty approximatively compact subset of X and $P : X \to 2^C$ a metric projection of X onto C defined by $P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}$.

Then $P_C(x) \neq \emptyset$, closed subset of C. In fact, $P_C(x)$ is compact.

The following is due to Rao and Mariadoss [9].

Let X be a Banach space and $f : X \to X$ a continuous map.Let C be an approximatively compact subset of X and an f-invariant set. Further, let $fx_0 = x_0$, $x_0 \in X$ and f satisfy the following conditions:
$\|fx - fy\| \leq \alpha\|x - y\| + \beta(\|x - fx\| + \|y - fy\|) + \gamma(\|x - fy\| + \|y - fx\|)$
for α, β, γ positive with $\alpha + 2\beta + 2\gamma \leq 1$, for all $x, y \in X$.
If the set of best C-approximants to x_0 is nonempty starshaped then it has an f-invariant point closest to x_0.

Note

It is easy to derive that f satisfies the condition $\|fx - fx_0\| \leq \|x - x_0\|$ for all $x \in X$, $(fx_0 = x_0)$, since $\alpha + 2\beta + 2\gamma \leq 1$, and since C is approximatively compact so the set of best C-approximants to x is compact.

Recently Carbone [3] proved the following:

THEOREM C — *Let $f : X \to X$ be a mapping on a Banach space X satisfying the following condition:* $\|fx - fy\| \leq \alpha\|x - y\| + \beta(\|x - fx\| + \|y - fy\|) + \gamma(\|x - fy\| + \|y - fx\|)$
for α, β, γ positive with $\alpha + 2\beta + 2\gamma \leq 1$.

Let $fx_0 = x_0$, for $x_0 \in X$ and C a subset of X such that $f(\partial C) \subseteq C$. If the set D of best C-approximants to x_0 is closed, strashaped and $\overline{f(D)}$ is compact with f continuous on D, then f has a fixed point closest to x_0.

We give the following to derive [9] as a special case of [3].

Remarks

1. f need not be continuous on the whole space X. Continuity only on D serves the purpose.

2. If C is approximatively compact then D is compact and $f(D)$ is compact since a continuous image of a compact set is compact.
3. C need not be f-invariant. Only $f(\partial C) \subseteq C$ is required.

In this section results in a locally convex Hausdorff topological vector space E are given.

Let C be a nonempty subset of E and let p be a continuous seminorm on E. For $x \in E$ define

$$d_p(x, C) = inf\{p(x-y) : y \in C\}$$

and

$$P_C(x) = \{y \in C : p(x-y) = d_p(x, C)\}.$$

A mapping $f : C \to C$ is said to be p-nonexpansive if for all x, $y \in C$

$$p(fx - fy) \leq p(x-y), \quad p \in \mathbf{P}$$

($\mathbf{P}$: family of continuous seminorms); $f : C \to C$ is asymptotically nonexpansive if there exists $\{k_n\}$ with $k_n \to 1$ as $n \to \infty$ such that

$$p(f^n x - f^n y) \leq k_n p(x-y)$$

for all x, $y \in C$, $p \in \mathbf{P}$, $n = 1, 2, \ldots\ldots$ It is assumed that $k_n \geq 1$ and $k_n \geq k_{n+1}$ for $n = 1, 2, \ldots\ldots$ [17].

f is uniformly asymptotically regular if for each $p \in \mathbf{P}$ and $\eta > 0$ there exists a $N(p, \eta)$ (=N say) such that

$$p(f^n x - f^{n+1} x) < \eta$$

for all $n \geq N$ and $x \in C$.

Vijayaraju [18] proved the following:

THEOREM V — *Let C be a nonempty subset of E.Let $x_0 \in E$ and $f : C \to C$ an asymptotically nonexpansive, uniformly asymptotically regular map. Suppose that the set D of best C-approximants to x_0 is nonempty, weakly compact, starshaped and invariant under f. If, further, $I - f$ is demiclosed then f has a fixed point closest to x_0.*

For further results on fixed points one should refer to [14], [16].

We give the following in locally convex Hausdorff topological vector space E improving Theorem V.

THEOREM 3 — *Let C be a subset of E, $fx_0 = x_0$, $x_0 \in E$ and $f(\partial C) \subseteq C$. Let $f : E \to E$ be an asymptotically nonexpansive uniformly asymptotically regular map. Assume that the set D of best C-approximants to x_0 is nonempty weakly compact, starshaped and invariant under f. If $I - f$ is demiclosed then f has a fixed point closest to x_0.*

Proof is on the same lines as in [18].

Note

We do not need $f : C \to C$ as is given in [18], only $f(\partial C) \subseteq C$ serves the purpose. Further related results are also given in [12].

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