Internat. J. Math. & Math. Sci.
VOL. 17 NO. 3 (1994) 441-446

REGULAR MEASURES AND NORMAL LATTICES

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(Received August 26, 1992 and in revised form December 27, 1992)

ABSTRACT. In this paper, we investigate $M(\mathcal{L})$ in case $\mathcal{L}$ is a normal lattice of subsets of X and we extend the results to $\mathcal{L}_1$, $\mathcal{L}_2$-lattices of subsets of X , such that $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mathcal{L}_1$ separates $\mathcal{L}_2$. We define the outer measures μ' and μ'' which prove very useful in proving some of the above results.

KEY WORDS AND PHRASES. Normal lattice, outer measure, separation.
1991 AMS SUBJECT CLASSIFICATION CODES. 28.

1. INTRODUCTION.

Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X. If $\mathcal{L}$ is a normal lattice, then in previous papers consequences pertaining to $I(\mathcal{L})$-the set of non-trivial finitely additive zero-one valued measures on $\mathcal{A}(\mathcal{L})$, the algebra generated by $\mathcal{L}$ have been investigated.

In the first part of this paper, we extend these results to $M(\mathcal{L})$, the set of non-trivial, non-negative finite and finitely additive measures on $\mathcal{A}(\mathcal{L})$. We extend these considerations to $\mathcal{L}_1, \mathcal{L}_2$-lattices of subsets of X such that $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mathcal{L}_1$ separates $\mathcal{L}_2$.

If $\mu \in M(\mathcal{L})$ then an auxiliary finitely subadditive outer measure μ' is associated with it and proves very useful in proving some of the above results. This along with another outer measure μ'' is considered in detail in the second part of the paper. It is shown that although μ' might not be a regular finitely subadditive outer measure, it is still true that an arbitrary set $E \subset X$ is μ'-measurable if and only if it splits just X additively. We note that if $\mu \in I(\mathcal{L})$ then μ' is clearly regular, but this need not be the case for $\mu \in M(\mathcal{L})$.

We begin with some standard background material (see also [1] and [5]) for the reader's convenience. Some related material can be found in [2], [3], and [4].

2. BACKGROUND AND NOTATIONS.

Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X. It is assumed that $\theta, X \in \mathcal{L}$. The lattice $\mathcal{L}$ is called *normal* if for any $L_1, L_2 \in \mathcal{L}$ with $L_1 \cap L_2 = \theta$ there exist $L_3, L_4 \in \mathcal{L}$ with $L_1 \subset L_3'$, $L_2 \subset L_4'$ and $L_3' \cap L_4' = \theta$ (where prime denotes complement). $\mathcal{L}$ is *almost countably compact* if $\mu \in I_R(\mathcal{L}')$ implies $\mu \in I_\sigma(\mathcal{L})$.

We give now some measure terminology. $M(\mathcal{L})$ denotes the set of finite valued bounded finitely additive measures on $\mathcal{A}(\mathcal{L})$. Without loss of generality may assume throughout that all measures are non-negative. A measure $\mu \in M(\mathcal{L})$ is called:

σ-smooth on $\mathcal{L}$ if for all sequences $\{L_n\}$ of sets of $\mathcal{L}$ with $L_n \downarrow \theta$, $\mu(L_n) \to 0$.

σ-smooth on $\mathcal{A}(\mathcal{L})$ if for all sequences $\{A_n\}$ of sets of $\mathcal{A}(\mathcal{L})$ with $A_n \downarrow \theta$, $\mu(A_n) \to 0$.

$\mathcal{L}$-regular if for any $A \in \mathcal{A}(\mathcal{L})$, $\mu(A) = sup\{\mu(L)/L \subset A, L \in \mathcal{L}\}$.

We denote by $M_R(\mathcal{L})$ the set of $\mathcal{L}$-regular measures of $M(\mathcal{L})$; $M_\sigma(\mathcal{L})$ the set of σ-smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$; $M^\sigma(\mathcal{L})$ the set of σ-smooth measures on $\mathcal{A}(\mathcal{L})$ of $M(\mathcal{L})$; $M_R^\sigma(\mathcal{L})$ the set of $\mathcal{L}$-regular measures of $M^\sigma(\mathcal{L})$.

In addition, $I(\mathcal{L}), I_R(\mathcal{L}), I_\sigma(\mathcal{L}), I^\sigma(\mathcal{L})$ and $I_R^\sigma(\mathcal{L})$ are the subsets of the corresponding M's which consist of the non-trivial zero-one valued measures.

Finally, for $\mathcal{L}_1 \subset \mathcal{L}_2$ two lattices of subsets of X, $\mathcal{L}_1$ *separates* $\mathcal{L}_2$ if $A, B \in \mathcal{L}_2$ and $A \cap B = \theta$ implies there exist $C, D \in \mathcal{L}_1$ such that $A \subset C, B \subset D$ and $C \cap D = \theta$.

3. NORMAL LATTICES.

THEOREM 1. Let $\mathcal{L}$ be normal and let $\mu \in M(\mathcal{L})$, $\nu \in M_R(\mathcal{L})$ with $\mu \le \nu$ on $\mathcal{L}$ and $\mu(X) = \nu(X)$. Then $\nu(L') = sup\{\mu(\widetilde{L}), \widetilde{L} \subset L', L, \widetilde{L} \in \mathcal{L}\}$.

PROOF. Let $L \in \mathcal{L}$ and $\varepsilon > 0$. Since $\nu \in M_R(\mathcal{L})$, there exists $A \subset L', A \in \mathcal{L}$ such that $\nu(L') - \varepsilon < \nu(A)$. Since $\mathcal{L}$ is normal, there exist $B, C \in \mathcal{L}$ such that $A \subset B' \subset C \subset L'$. Hence,

$$\nu(A) \le \nu(B') \le \mu(B') \le \mu(C) \le \nu(C) \le \nu(L')$$

and then $\nu(L') - \mu(C) \le \nu(L') - \nu(A) < \varepsilon$, i.e., $\nu(L') < \mu(C) + \varepsilon$, ε arbitrary small. Therefore, $\nu(L') = sup\{\mu(\widetilde{L}), \widetilde{L} \subset L', \widetilde{L} \in \mathcal{L}\}$.

DEFINITION 1. Let $\mu \in M(\mathcal{L})$ and define $\mu'(E) = inf\{\sum_{i=1}^{n} \mu(L_i'), E \subset \bigcup_{i=1}^{n} L_i', L_i \in \mathcal{L}, E \subset X.\}$ $= inf\{\mu(L'), E \subset L', L \in \mathcal{L}\}$.

THEOREM 2. Suppose $\mathcal{L}$ is normal and let $\mu \in M(\mathcal{L}), \nu \in M_R(\mathcal{L})$ with $\mu \le \nu$ on $\mathcal{L}$ and $\mu(X) = \nu(X)$. Then $\mu \le \nu = \nu' = \mu'$ on $\mathcal{L}$.

PROOF. If $A \in \mathcal{L}, \mu(A) \le \mu(L')$ for all $A \subset L', L \in \mathcal{L}$. Therefore $\mu(A) \le \mu'(A) = inf\ \mu(L')$, $A \subset L'$, i.e., $\mu \le \mu'$ on $\mathcal{L}$. For $\nu \in M_R(\mathcal{L})$ we have for any $A \in \mathcal{L}$:

$$\nu(A) = inf\ \{\nu(L')/A \subset L', L \in \mathcal{L}\} = \nu'(A)$$

i.e., $\nu = \nu'$ on $\mathcal{L}$. $\mu \le \nu$ on $\mathcal{L}$ and $\mu(X) = \nu(X)$ implies $\mu \ge \nu$ on $\mathcal{L}'$. Hence $\nu'(A) = inf \nu(L')$ $\le inf \mu(L') = \mu'(A), A \subset L'$, i.e., $\nu' \le \mu'$ on $\mathcal{L}$. To show that $\nu' = \mu'$ on $\mathcal{L}$, suppose that $\nu' \ne \mu'$ on $\mathcal{L}$. Then there exists $L \in \mathcal{L}$ such that $\nu'(L) < \mu'(L)$. Since $\nu \in M_R(\mathcal{L})$, there exists $A \in \mathcal{L}, L \subset A'$ such that $\nu(A') - \varepsilon < \nu(L) = \nu'(L)$.

$\mathcal{L}$ being normal, there exist $C, D \in \mathcal{L}$ such that $L \subset C' \subset D \subset A'$ and $\mu'(L) \le \mu'(C')$ $\le \mu(C') \le \mu(D)$. But $\mu(D) \le \nu(D) \le \nu(A') < \nu'(L) + \varepsilon < \mu'(L) + \varepsilon \le \mu(D) + \varepsilon$, contradiction.

THEOREM 3. Let $\mathcal{L}$ be normal and let $\mu \in M_\sigma(\mathcal{L})$, $\nu \in M_R(\mathcal{L})$ with $\mu \le \nu$ on $\mathcal{L}$ and $\mu(X) = \nu(X)$. Then $\nu \in M_\sigma(\mathcal{L}')$.

PROOF. Let $L_n' \downarrow \theta, L_n \in \mathcal{L}$. Since $\nu \in M_R(\mathcal{L})$ and $\mu \in M_\sigma(\mathcal{L})$ with $\mu \le \nu$ on $\mathcal{L}$, $\nu(L') = sup$ $\mu(\widetilde{L}), \widetilde{L} \subset L'; L, \widetilde{L} \in \mathcal{L}$. Given $\varepsilon > 0$, there exist $\tilde{L}_n \subset L_n', \tilde{L}_n \in \mathcal{L}$ such that $\nu(L_n') < \mu(\tilde{L}_n) + \frac{\varepsilon}{2}$. May assume $\tilde{L}_n \downarrow$ and since $\tilde{L}_n \subset L_n' \downarrow \theta$ it follows $\tilde{L}_n \downarrow \theta$. Since $\mu \in M_\sigma(\mathcal{L}), \mu(\tilde{L}_n) < \frac{\varepsilon}{2}$, all $n \ge N(\varepsilon)$. Hence $\nu(L_n') < \varepsilon$ for all $n \ge N(\varepsilon)$, i.e., $\nu \in M_\sigma(\mathcal{L}')$.

THEOREM 4. Suppose $\mathcal{L}$ is normal and let $\mu \in M(\mathcal{L}), \nu_1, \nu_2 \in M_R(\mathcal{L})$ with $\mu \le \nu_1$ on $\mathcal{L}$, $\mu \le \nu_2$ on $\mathcal{L}$ and $\mu(X) = \nu_1(X) = \nu_2(X)$. Then $\nu_1 = \nu_2$.

PROOF. By Theorem 2,

$$\mu \le \nu_1 = \nu_1' = \mu', \quad \mu \le \nu_2 = \nu_2' = \mu' \text{ on } \mathcal{L}.$$

Hence $\nu_1 = \nu_2 = \mu'$

THEOREM 5. Let $\mathcal{L}$ be almost countably compact and let $\mu \in M_R(\mathcal{L}')$. Then $\mu \in M_\sigma(\mathcal{L})$.

PROOF. Suppose $\mu \notin M_\sigma(\mathcal{L})$. Then there exists $A_n \downarrow \theta, A_n \in \mathcal{L}$ with $\mu(A_n) \not\to 0$. Since $\mu \in M_R(\mathcal{L}')$, there exists $B_n \in \mathcal{L}, A_n \supset B'_n$ and $0 \not\leftarrow \mu(A_n) \sim \mu(B'_n)$. May assume $B'_n \neq \theta$ all n and $B'_n \downarrow \emptyset$, hence $\{B'_n\}$ has the finite intersection property. Therefore there exists $\lambda \in I_R(\mathcal{L}')$ such that $\lambda(B'_n) = 1$ all n. Since $A_n \supset B'_n$ it follows $\lambda(A_n) = 1$ all n. But $A_n \downarrow \theta$; then $\lambda \notin I_\sigma(\mathcal{L})$, contradiction since $\mathcal{L}$ is almost countably compact.

THEOREM 6. Let $\mathcal{L}_1 \subset \mathcal{L}_2$ and suppose that $\mathcal{L}_1$ separates $\mathcal{L}_2$. Let $\mu \in M_R(\mathcal{L}_1)$ and consider the extension $\nu \in M_R(\mathcal{L}_2)$. Then:

(a) ν is $\mathcal{L}_1$-regular on $\mathcal{L}'_2$.

(b) If $\mu \in M^\sigma_R(\mathcal{L}_1)$ then $\nu \in M_\sigma(\mathcal{L}'_2)$.

(c) ν is unique.

PROOF.

(a) Define $\mu_i(L_2) = inf\ \mu(L'_1)$, $L_2 \subset L'_1, L_2 \in \mathcal{L}_2, L_1 \in \mathcal{L}_1$. For any $L'_1 \supset L_2$ we have $\nu(L_2) \leq inf\ \mu(L'_1) = \mu_i(L_2)$, hence $\nu \leq \mu_i$ on $\mathcal{L}_2$. Suppose that $\nu(L_2) < \mu_i(L_2)$ for some $L_2 \in \mathcal{L}_2$. Since $\nu \in M_R(\mathcal{L}_2)$, there exists $\tilde{L}_2 \in \mathcal{L}_2$, $L_2 \subset \tilde{L}'_2)$ such that $\nu(\tilde{L}'_2) < \nu(L_2) + \varepsilon$. By separation, there exist $L_1, \tilde{L}_1 \in \mathcal{L}_1$ such that $L_2 \subset L_1, \tilde{L}_2 \subset \tilde{L}_1$ and $L_1 \cap \tilde{L}_1 = \emptyset$. Then $L_2 \subset L_1 \subset \tilde{L}'_1 \subset \tilde{L}'_2$ and $\nu(L_2) < \nu(L_1) = \mu(L_1) < \nu(\tilde{L}'_2) < \mu_i(L_2) + \varepsilon$, ε arbitrary small. It follows $\mu(L_1) < \mu_i(L_2)$. But $L_2 \subset L_1$ implies $\mu_i(L_2) \leq \mu(L_1)$, contradiction. Hence we must have $\nu = \mu_i$ on $\mathcal{L}_2$ or $\nu = \mu_*$ on $\mathcal{L}'_2$.

(b) Let $L'_n \in \mathcal{L}'_2, L'_n \downarrow \theta$. $\nu(L'_n) = \mu_*(L'_n) = sup\{\mu(\tilde{L}_n), \tilde{L}_n \subset L'_n, \tilde{L}_n \in \mathcal{L}_1\}$. Since $L'_n \downarrow \theta$ and $\tilde{L}_n \subset L'_n$, may assume $\tilde{L}_n \downarrow \theta$. Given $\varepsilon > 0$, there exists $\tilde{L}_n \subset L'_n$ such that $\nu(L'_n) - \varepsilon < \mu(\tilde{L}_n)$. $\mu \in M^\sigma_R(\mathcal{L}_1)$ implies $\mu(\tilde{L}_n) \to 0$, hence $\nu(L'_n) \to 0$, i.e., $\nu \in M_\sigma(\mathcal{L}'_2)$.

(c) Suppose for $\mu \in M^\sigma_R(\mathcal{L}_1)$ there are two extensions $\nu_1, \nu_2 \in M_R(\mathcal{L}_2)$. By (a) ν_1, ν_2, are $\mathcal{L}_1$-regular on $\mathcal{L}'_2$, i.e.,

$\nu_1(L'_2) = sup\ \mu(L_1), L_1 \subset L'_2, L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2$.

$\nu_2(L'_2) = sup\ \mu(L_1), L_1 \subset L'_2, L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2$.

Hence $\nu_1(L'_2) = \nu_2(L'_2)$ and then $\nu_1 = \nu_2$ on $\mathcal{L}'_2$ which implies $\nu_1 = \nu_2$ on $\mathcal{L}_2$.

THEOREM 7. Let $\mathcal{L}_1 \subset \mathcal{L}_2$ and suppose $\mathcal{L}_1$ separates $\mathcal{L}_2$. Let $\mu \in M(\mathcal{L}_1), \nu \in M_R(\mathcal{L}_1)$ with $\mu \leq \nu$ on $\mathcal{L}_1$ and $\mu(X) = \nu(X)$. Consider the extensions $\tau \in M(\mathcal{L}_2), \tau/_{\mathcal{A}(\mathcal{L}_1)} = \mu$ and $\lambda \in M_R(\mathcal{L}_2)$, $\lambda/_{a(\mathcal{L}_1)} = \nu$. Then $\tau \leq \lambda$ on $\mathcal{L}_2$.

PROOF. Let $L_2 \in \mathcal{L}_2$, arbitrary. Since λ is $\mathcal{L}_2$-regular, given $\varepsilon > 0$, there exists $\tilde{L}_2 \in \mathcal{L}_2$, $L_2 \subset \tilde{L}'_2$ such that $\lambda(\tilde{L}'_2) < \lambda(L_2) + \varepsilon$. By separation, there exists $L_1, \tilde{L}_1 \in \mathcal{L}_1$ such that $L_2 \subset L_1$, $\tilde{L}_2 \subset \tilde{L}_1$ and $L_1 \cap \tilde{L}_1 = \theta$. Therefore, we have $L_2 \subset L_1 \subset \tilde{L}'_1 \subset \tilde{L}'_2$ and $\tau(L_2) \leq \tau/_{\mathcal{A}(\mathcal{L}_1)}(L_1) = \mu(L_1) \leq \nu(L_1) \leq \nu(\tilde{L}'_1) = \lambda/_{\mathcal{A}(\mathcal{L}_1)}(\tilde{L}'_1) \leq \lambda(\tilde{L}'_2) < \lambda(L_2) + \varepsilon$ and $\lambda = \nu$ on $\mathcal{L}_1, \mathcal{L}'_1$; ε arbitrary small implies $\tau(L_2) \leq \lambda(L_2)$. L_2 arbitrary in $\mathcal{L}_2$ shows that $\tau \leq \lambda$ on $\mathcal{L}_2$.

THEOREM 8. Let $\mathcal{L}$ be normal and almost countably compact. Then $M_R(\mathcal{L}) \subset M_\sigma(\mathcal{L}')$.

PROOF. Let $\mu \in M_R(\mathcal{L})$. Then $\mu \leq \rho \in M_R(\mathcal{L}')$ on $\mathcal{L}'$ and $\mu(X) = \rho(X)$. Hence $\rho \leq \mu$ on $\mathcal{L}$ and $\rho \in M_\sigma(\mathcal{L})$ since $\mathcal{L}$ is almost countably compact. But $\rho \in M_\sigma(\mathcal{L})$ and $\mu \in M_R(\mathcal{L})$ and $\rho \leq \mu$ on $\mathcal{L}$, therefore, $\mathcal{L}$ being normal it follows $\mu \in M_\sigma(\mathcal{L}')$.

4. SOME FINITELY SUBADDITIVE OUTER MEASURES.

DEFINITION 2. μ defined on $\mathcal{P}(X)$ is a *finitely subadditive outer* measure if:

(a) μ is nondecreasing;

(b) $\mu(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} \mu(E_i)$, for any $E_1, E_2, \cdots E_n \in X$;

(c) $\mu(\theta) = 0$

Clearly, μ', as defined in Section 3, is a finitely subadditive outer measure.

Let $\mathcal{S}_{\mu'}$ be the set of all μ'-measurable sets, where E is measurable with respect to μ' if for any $A \subset X$, $\mu'(A) = \mu'(A \cap E) + \mu'(A \cap E')$

THEOREM 9. $E \subset \mathcal{S}_{\mu'}$ if and only if $\mu'(A') \geq \mu'(A' \cap E) + \mu'(A' \cap E')$ for all $A \in \mathcal{L}$.

PROOF.

(a) If $E \subset \mathcal{S}_{\mu'}$, then the relation clearly holds.

(b) Let B be arbitrary in $X, B \subset A', A \in \mathcal{L}$. We have:

$$\mu(A') = \mu'(A') \geq \mu'(A' \cap E) + \mu'(A' \cap E') \geq \mu'(B \cap E) + \mu'(B \cap E'),$$

by assumption and μ' being monotone. Since this is true for any $B \subset A', A \in \mathcal{L}$, it follows $\mu'(B) \geq \mu'(B \cap E) + \mu'(B \cap E')$. By the definition of μ' as an outer measure, we have for $B = (B \cap E) U (B \cap E')$: $\mu'(B) \leq \mu'(B \cap E) + \mu'(B \cap E')$. Therefore, $\mu'(B) = \mu'(B \cap E) + \mu'(B \cap E')$, B arbitrary in X, i.e., $E \in \mathcal{S}_{\mu'}$.

DEFINITION 3. Let $\mu \in M(\mathcal{L})$ and define the inner measure $\mu_i(E) = sup \mu(L)$, $L \subset E, L \in \mathcal{L}, E \subset X$.

THEOREM 10. The following statements are true:

(a) $\mu(X) = \mu_i(L) + \mu'(L'), L \in \mathcal{L}$.

(b) $\mu(X) = \mu_i(L') + \mu'(L)$

PROOF. Clear.

DEFINITION 4. Let $\mu \in M_\sigma(\mathcal{L}), E \subset X$ and define $\mu''(E) = inf \sum_{i=1}^{\infty} \mu(L_i'), E \subset \bigcup_{i=1}^{\infty} L_i', L_i \in \mathcal{L}$. Let $\mathcal{S}_{\mu''}$ be the set of μ''-measurable sets, where E is measurable with respect to μ'' if for any $A \subset X \mu''(A) = \mu''(A \cap E) + \mu''(A \cap E')$.

THEOREM 11. μ'' is an outer measure.

PROOF. Clear.

THEOREM 12. $E \in \mathcal{S}_{\mu''}$ if and only if $\mu''(A') \geq \mu''(A' \cap E) + \mu''(A' \cap E')$ for all $A \in \mathcal{L}$.

PROOF.

(a) If $E \in \mathcal{S}_{\mu''}$, then clearly $\mu''(A') \geq \mu''(A' \cap E) + \mu''(A' \cap E')$ for $A \in \mathcal{L}$.

(b) Let B be arbitrary set in X and let $B \subset \bigcup_{i=1}^{\infty} L_i', L_i \in \mathcal{L}$ all i. Then, since $\mu'' \leq \mu$ on $\mathcal{L}'$, we have

$$\sum_{i=1}^{\infty} \mu(L_i') \geq \sum_{i=1}^{\infty} \mu''(L_i') \geq \sum_{i=1}^{\infty} [\mu''(L_i' \cap E) + \mu''(L_i' \cap E')] = \sum_{i=1}^{\infty} \mu''(L_i' \cap E) +$$

$$\sum_{i=1}^{\infty} \mu''(L_i' \cap E') \geq \mu''(\bigcup_{i=1}^{\infty} L_i' \cap E) + \mu''(\bigcup_{i=1}^{\infty} L_i' \cap E') \geq \mu''(B \cap E) + \mu''(B \cap E').$$

This holds for all $B, B \subset \bigcup_{i=1}^{\infty} L_i'$, therefore $\mu''(B) = inf \sum_{i=1}^{\infty} \mu(L_i') \geq \mu''(B \cap E) + \mu''(B \cap E')$ and since B was arbitrary in X it follows $E \in \mathcal{S}_{\mu''}$.

THEOREM 13. Let $\mu \in M(\mathcal{L}), E \subset X$. $E \in \mathcal{S}_{\mu'}$ iff $\mu_i(E) = \mu'(E)$.

PROOF. Suppose $E \in \mathcal{S}_{\mu'}$. Then $\mu(X) = \mu'(E') + \mu'(E)$. By Theorem 10, $\mu(X) = \mu_i(E') + \mu'(E)$, so we have $\mu'(E') = \mu_i(E')$. Hence $\mu(X) - \mu'(E') = \mu(X) - \mu_i(E')$ and then $\mu_i(E) = \mu'(E)$. Conversely, suppose $\mu_i(E) = \mu'(E)$. Then, given $\varepsilon > 0$, there exists $\widehat{L} \in \mathcal{L}, \widehat{L} \subset E$ and $\mu(\widehat{L}) + \frac{\varepsilon}{2} > \mu(E)$. Also, be definition of μ', there exists $\widetilde{L} \in \mathcal{L}$ such that $\tilde{L}' \supset E \supset \widehat{L}$ and $\mu(\tilde{L}') < \mu'(E) + \frac{\varepsilon}{2}$. Now, let $A' \in \mathcal{L}'$. Then $\mu'(A' \cap E) \leq \mu'(A' \cap \tilde{L}') = \mu(A') + \mu(\tilde{L}') - \mu(A' U \tilde{L}')$ $\leq \mu(A') + \mu'(E) + \frac{\varepsilon}{2} - \mu(A' U \widehat{L}) \leq \mu(A') + \mu(\widehat{L}) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} - \mu(A' U \widehat{L}) = \mu(A' \cap \widehat{L}) + \varepsilon$. Now $E' \subset \widehat{L}'$, hence $A' \cap E' \subset A' \cap \widehat{L}'$. Thus $\mu'(A' \cap E) + \mu'(A' \cap E') \leq \mu(A' \cap \widehat{L}) + \varepsilon + \mu'(A' \cap \widehat{L}') = \mu(A' \cap \widehat{L})$

$+ \varepsilon + \mu(A' \cap \widehat{L}') = \mu(A') + \varepsilon$, ε arbitrary.
Therefore, $\mu'(A' \cap E) + \mu'(A' \cap E') \leq \mu(A') = \mu'(A'), A' \in \mathcal{L}'$, i.e., $E \in \mathcal{S}_{\mu'}$.

THEOREM 14. Let $\mu \in M_\sigma(\mathcal{L})$ and define $\mu''(E)$ as above. Then

(a) $\mu''(X) = \mu(X)$

(b) $\mu \leq \mu''$ on $\mathcal{L}$.

PROOF.

(a) If $\mu''(X) \neq \mu(X)$, there exists $L_i \in \mathcal{L}, i = 1,2, \cdots$ such that $X = \bigcup_{i=1}^{\infty} L_i'$ and $\sum_{i=1}^{\infty} \mu(L_i') < \mu(X)$. But $\mu(L_i' = lim \sum_{i=1}^{n} \mu(L_i') \geq \lim_n \mu(\bigcup_{i=1}^{n} L_i')$, $\bigcup_{i=1}^{n} L_i' \uparrow$ and $\bigcup_{i=1}^{n} L_i' \in \mathcal{L}'$ and $\bigcup_{i=1}^{n} L_i' \uparrow \bigcup_{i=1}^{\infty} L_i' = X$. Therefore, since $\mu \in M_\sigma(\mathcal{L})$ we have $\lim_n \mu(\bigcup_{i=1}^{n} L_i') = \mu(X)$, contradiction.

(b) Suppose there exists $L \in \mathcal{L}$ such that $\mu(L) > \mu''(L)$. Then $\mu''(X) \leq \mu''(L) + \mu''(L')$ $\leq \mu''(L) + \mu(L') < \mu(L) + \mu(L') = \mu(X)$ which contradicts part (a) of the theorem.

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