Internat. J. Math. & Math. Sci.
VOL. 18 NO. 2 (1995) 391-396

THE LAW OF THE ITERATED LOGARITHM FOR EXCHANGEABLE RANDOM VARIABLES

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(Received January 13, 1993 and in revised form July 25, 1993)

ABSTRACT. In this note, necessary and sufficient conditions for laws of the iterated logarithm are developed for exchangeable random variables.

KEY WORDS AND PHRASES. The law of the iterated logarithm, exchangeable sequence of random variables.

1991 AMS SUBJECT CLASSIFICATION CODES. Primary, 60G09, 60F15; Secondary, 60G07.

1. INTRODUCTION.

In 1929, Kolmogorov proved a law of the iterated logarithm (LIL) for independent random variables under certain boundedness conditions. Hartman and Winter in 1941 verified that the LIL is universally true for i.i.d. random variables when the second moment exists. There are certain extensions of the LIL to martingales. However, there appears to have been no discussions on this problem for exchangeable random variables. We address this problem in this paper and extend the LIL to exchangeable random variables with necessary and sufficient conditions for the LIL in terms of conditional mean and variance.

Random variables (r.v.'s) $X_1, \cdots, X_n$ are said to be exchangeable if the joint distribution of $X_1, \cdots, X_n$ is permutation invariant. A sequence of r.v.'s $\{X_n\}$ is said to be exchangeable if every finite subset of the sequence is exchangeable. Obviously, i.i.d. random variables are exchangeable, but not vice versa. The LIL is said to hold for a sequence of r.v.'s $\{X_n\}$ with $EX_n = 0$ for all n if

$$P\left[\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{\sqrt{2S_n^2 \log\log S_n^2}} = 1\right] = 1$$

where $S_n^2 = \sum_{i=1}^{n} EX_i^2$ and log denote the natural log to the base e. The following example shows that the LIL can fail even for exchangeable r.v.'s while a sequence of exchangeable r.v.'s may satisfy the LIL and not be independent r.v.'s.

EXAMPLE 1. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables such that $EX_1 = 0$ and $EX_1^2 = 1$ and let $Y_n = ZX_n$, $n \geq 1$, where the random variable Z is independent of the sequence $\{X_n, n \geq 1\}$ with $P(Z = a) = P(Z = b) = 0.5$. It is not difficult to see that $\{Y_n\}$ is a sequence of exchangeable r.v.'s.

If $a = 2$ and $b = 0$, then $EY_n = 0$ and $EY_n^2 = 2$ for every $n \geq 1$. We define $S_n^2 = \sum_1^n EY_j^2$ and $U_n^2 = 2\ loglog\ S_n^2$. Clearly,

$$P\left(\limsup_{n\to\infty} \frac{\sum_1^n Y_j}{S_n U_n} = 1\right) = 0.5P\left(\limsup_{n\to\infty} \frac{\sum_1^n X_j}{\sqrt{2n\ loglog\ n}} = \frac{1}{\sqrt{2}}\right) = 0 \tag{1.1}$$

in view of the fact that by [4],

$$\limsup_{n\to\infty} \frac{\sum_1^n X_j}{\sqrt{2n\ loglog\ n}} = \limsup_{n\to\infty} \frac{\sum_1^n (-X_j)}{\sqrt{2n\ loglog\ n}} = 1, \text{ a.s.} \tag{1.2}$$

In this case, the LIL is almost nowhere true for the sequence of exchangeable random variables $\{Y_n\}$ versus the LIL holding for the sequence of i.i.d. random variables $\{X_n\}$.

However, if $a = 1$ and $b = -1$, then $EY_n^2 = 1$, $S_n^2 = n$, and $U_n^2 = 2\ loglog\ n$, which yields from (1.2)

$$P\left(\limsup_{n\to\infty} \sum_1^n Y_j/S_nU_n = 1\right)$$

$$= 0.5\ P\left(\limsup_{n\to\infty} \sum_1^n X_j/\sqrt{2n\ loglog\ n} = 1\right)$$

$$+ 0.5P\left(\limsup_{n\to\infty} \sum_1^n (-X_j)/\sqrt{2n\ loglog\ n} = 1\right) = 1. \tag{1.3}$$

This is another case where the LIL holds for exchangeable r.v.'s$\{Y_n\ , n \geq 1\}$ which might definitely not be a sequence of independent r.v.'s as long as

$$P(X_1 < a)P(X_1 < b) + P(X_1 > -a)P(X_1 > -b)$$
$$\neq P(X_1 < a)P(X_1 > -a) + P(X_1 < b)P(X_1 > -b).$$

A similar example can be constructed to show that under certain conditions the LIL holds for martingales but fails for exchangeable r.v.'s and vice versa. Thus, conditions for the LIL to hold may be very different for exchangeable r.v.'s than for independent r.v.'s or martingales.

Necessary and sufficient conditions for the LIL to hold for exchangeable r.v.'s are established in the next section.

2. THE LIL FOR EXCHANGEABLE r.v.'s.

Below we establish the LIL and give the necessary and sufficient conditions for exchangeable r.v.'s to satisfy the LIL by using de Finetti's theorem. Let Φ denote the collection of distribution functions on $\Re$ (real numbers) and provide Φ with topology of weak convergence of distribution functions. Then, de Finetti's theorem [2] asserts that for an infinite sequence of exchangeable r.v.'s $\{X_n\}$ there exists a probability measure μ on the Borel σ-field Σ of subsets of Φ such that

$$P\{g(X_1,\cdots,X_n)\in B\} = \int_\Phi P_F\{\ g[X_1,\cdots,X_n]\in B\}\ d\mu(F) \tag{1.4}$$

for any $B \in \mathfrak{B}$ and any Borel function $g: R^n \to R, n \geq 1$. Moreover, $P_F[g(X_1,\cdots,X_n) \in B]$ is computed under the assumption that the sequence of r.v.'s $\{X_n\}$ is i.i.d. with common distribution function F, where $E_F g(X_n)$ is the conditional mean obtained by integrating $g(x)$ with respect to P_F ( ) given by (1.4).

From (1.4), we know that if $\{X_n\}$ is a sequence of exchangeable r.v.'s on $(\Omega, \mathcal{A}, P)$, then $\{E_F g(X_n)\}$ is a sequence of random variables on (Φ, Σ, μ) and for each $F \in \Phi$ given, $\{X_n\}$ are independent, identically distributed.

Taylor and Hu (1987) showed that for a sequence of exchangeable r.v.'s $\{X_n\}$ such that $E_F\,|\,X_1\,| < \infty\ \mu-$a.s.

$$E_F X_1 = 0\ \mu-\text{a.s. if and only if } \tfrac{1}{n}\sum_{k=1}^{n} X_k \to 0 \text{ a.s.}$$

Moreover, it was observed that $E_F X_1 = 0$ μ-a.s. is equivalent to $E(X_1, X_2) = 0$. Blum, Chernoff, Rosenblatt, and Teicher (1958) showed that for a sequence of exchangeable r.v.'s $\{X_n\}$ such that $EX_1^2 < \infty$

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n} X_k \text{ converged in distribution to a } N(0,\sigma^2) \text{ r.v.}$$

if and only if

$$E_F X_1 = 0\ \mu\text{-a.s. and } E_F X_1^2 = \sigma^2\ \mu\text{-a.s.} \tag{1.5}$$

which is equivalent to the alternative and structurally simpler condition $EX_1X_2 = 0$ and $EX_1^2X_2^2 = 1$.

The necessary and sufficient conditions for the LIL to hold for exchangeable random variables are patterned after these results.

THEOREM 1. Let $\{X_n, n \geq 1\}$ be a sequence of exchangeable r.v.'s with $EX_1 = 0$ and $0 < EX_1^2 = \sigma^2 < \infty$. Then

$$\limsup_{n\to\infty} \sum_{1}^{n} X_j \,/\, \sqrt{2n\ \log\log n} = \sigma, \text{a.s.}, \tag{1.6}$$

if and only if

$$\nu_F = E_F X_1 = 0 \text{ and } \sigma_F^2 = E_F(X_1 - \nu_F)^2 = \sigma^2,\ \mu\text{-a.s.} \tag{1.7}$$

COMMENT. Condition (1.7) is equivalent to $EX_1X_2 = 0$ and $EX_1^2X_2^2 = 1$.

PROOF. First, observe that (1.6) is equivalent to

$$P\left[\sum_{1}^{n} X_j \,/\, \sqrt{2n\ \log\log n}\ \geq c\sigma, \text{i.o.}\right] = \begin{cases} 0, & \text{if } c > 1 \\ 1, & \text{if } c < 1 \end{cases} \tag{1.8}$$

Next, from (1.4) and by the continuity of probability measure and the bounded convergence theorem,

$$P\left[\sum_{1}^{n} X_j \,/\, \sqrt{2n\ \log\log n}\ \geq c\sigma, \text{i.o.}\right] \tag{1.9}$$

$$= \lim_{k\to\infty} \lim_{m\to\infty} P\left[\bigcup_{n=k}^{m}\left(\sum_{1}^{n} X_j \,/\, \sqrt{2n\ \log\log n}\ \geq c\sigma\right)\right]$$

$$= \lim_{k\to\infty}\lim_{m\to\infty}\int_\Phi P_F\left[\bigcup_{n=k}^{m}\left(\sum_1^n X_j \,/\, \sqrt{2n\ loglog\ n}\ \geq c\sigma\right)\right]d\mu(F)$$

$$= \int_\Phi \lim_{k\to\infty}\lim_{m\to\infty} P_F\left[\bigcup_{n=k}^{m}\left(\sum_1^n X_j \,/\, \sqrt{2n\ loglog\ n}\ \geq c\sigma\right)\right]d\mu(F)$$

$$= \int_\Phi P_F\left(\sum_1^n X_j \,/\, \sqrt{2n\ loglog\ n}\ \geq c\sigma,\ \text{i.o.}\right)d\mu(F)$$

$$= \int_\Phi P_F\left\{\sum_1^n (X_j-\nu_F) \,/\, \sqrt{2n\ loglog\ n}\ \geq c\sigma - \sqrt{n/(2\ loglog\ n)}\ \nu_F, \text{i.o.}\right\}d\mu(F).$$

Then, we conclude from (1.8) and (1.9) that (1.6) is equivalent to (1.10) and (1.11) where

$$P_F\left\{\sum_1^n (X_j-\nu_F) \,/\, \sqrt{2n\ loglog\ n}\ \geq c\sigma - \sqrt{n/(2\ loglog\ n)}\ \nu_F, \text{i.o.}\right\} \tag{1.10}$$

$= 0$, μ-a.s., for any $c > 1$.

and

$$P_F\left\{\sum_1^n (X_j-\nu_F) \,/\, \sqrt{2n\ loglog\ n}\ \geq c\sigma - \sqrt{n/(2\ loglog\ n)}\ \nu_F, \text{i.o.}\right\} \tag{1.11}$$

$= 1$, μ-a.s., for any $c < 1$.

Clearly, the "if" part follows easily from (1.10)-(1.11) since $\{X_n - \nu_F, n \geq 1\}$ are conditionally i.i.d. with zero mean given F, which leads to

$$P_F\left[\sum_1^n (X_j-\nu_F) \,/\, \sqrt{2n\ loglog\ n}\ \geq c\sigma_F\ ,\ \text{i.o.}\right] \tag{1.12}$$

$$= \begin{cases} 0 \text{ for any } c > 1 \\ 1 \text{ for any } c < 1 \end{cases}, \quad \text{for each } F \in \Phi,$$

when $0 < \sigma_F < \infty$ by the LIL. The above with $\nu_F = 0$ and $\sigma_F = \sigma$, μ-a.s., confirms (1.10) and (1.11) and hence establishes (1.6).

To prove the "only if" part, we first compare (1.10) with (1.12) to assert $\nu_F \leq 0$, μ-a.s.. Otherwise, if $\mu(F{:}\nu_F > 0) > 0$, there exists a $\varepsilon > 0$ such that

$$\mu(E) > 0,\ E = \{F{:}\nu_F > \varepsilon\} \tag{1.13}$$

and on the set E, for all sufficiently large n

$$c\sigma - \sqrt{n/(2\ loglog\ n)}\ \ \nu_F < -2\sigma_F. \tag{1.14}$$

Hence from (1.12),

$$P_F\left[\sum_1^n (X_j-\nu_F) \,/\, \sqrt{2n\ loglog\ n}\ \geq c\sigma - \sqrt{n/(2\ loglog\ n)}\ \ \nu_F, \text{i.o.}\right] \tag{1.15}$$

$$\geq P_F\left[\sum_1^n (X_j-\nu_F) \,/\, \sqrt{2n\ loglog\ n}\ \geq -2\sigma_F,\ \text{i.o.}\right]$$

$= 1$, for any $c > 1$ and $F \in E$.

It should be mentioned that although (1.15) is deduced under the assumption that $0 < \sigma_F < \infty$, (1.15) is still true when $\sigma_F = 0$ as a trivial case and $\sigma_F = \infty$ is excluded from consideration in view of the fact that $E\sigma_F^2 \leq E\ E_F X_1^2 = \sigma^2 < \infty$. The contradiction of (1.15) to (1.10) makes the assertion $\nu_F \leq 0, \mu$-a.s.. A similar argument from (1.11) and (1.12) concludes $\nu_F \geq 0$, μ-a.s., thus, $\nu_F = 0$, μ-a.s., has been confirmed. With this, we can reduce (1.10) and (1.11) to

$$P_F\left\{\sum_1^n (X_j - \nu_F) \,/\, \sqrt{2n\ loglog\ n} \geq c\sigma, \text{ i.o.}\right\} \tag{1.16}$$

$$= \begin{cases} 0 \text{ for any } c > 1 \\ 1 \text{ for any } c < 1 \end{cases}, \quad \mu\text{-a.s.},$$

A comparison of (1.12) with (1.16) yields $\sigma_F = \sigma$, μ a.s., to complete the proof of Theorem 1. □

We remark that the conditions of Theorem 1 are satisfied of $a = 1$ and $b = 1$, but are not satisfied if $a = 2$ and $b = 0$.

EXAMPLE 2. Let X be a random variable with $EX = 0$ and $0 < EX^2 < \infty$, and let $X_n = X, n \geq 1$. Then (1.7) and (1.6) clearly fail for the exchangeable sequence $\{X_n, n \geq 1\}$.

For a sequence of random variables $\{X_n, n \geq 1\}$, let T be the tail σ-field defined by $T = \bigcap_{n=1}^{\infty} \sigma(X_j: j \geq n)$ and let

$$T = \limsup_{n\to\infty} \sum_{j=1}^n X_j \,/\, \sqrt{2n\ loglog\ n}. \tag{1.17}$$

When $\{X_n, n \geq 1\}$ is a sequence of i.i.d. r.v.'s such that $EX_1 = 0$ and $EX_1^2 = \sigma^2$, T is almost surely equal to the constant σ. Theorem 1 also yields $T = \sigma$ a.s. if condition (1.6) holds for exchangeable random variables, and the example in Section 1 shows that Theorem 1 may be obtained for non-independent random variables. It is also worth observing that for exchangeable r.v.'s condition (1.7) is the necessary and sufficient condition for $n^{-1/2} \sum_{j=1}^n X_j$ to converge in distribution to a $N(0, \sigma^2)$ r.v. It is possible for $n^{-1/2} \sum_{j=1}^n X_j$ to converge in distribution to a mixture of normal distributed r.v.'s (cf: Chapter 2 of Taylor, Daffer, and Patterson). For example, if $\{X_n,\ n \geq 1\}$ is a sequence of exchangeable r.v.'s with $EX_1 = 0$, $EX_1^2 < \infty$, and $E(X_1, X_2) = 0$ (equivalently $\nu_F = 0$ μ-a.s.), then $n^{-1/2} \sum_{j=1}^n X_j$ converges in distribution to a r.v. Z which has distribution function $F(x) = \int_0^\infty \Phi(\sigma^{-1}x)dG(\sigma)$ where Φ is the standard normal distribution function and G is a distribution function with support contained in $[0, \infty)$. Theorem 2 provides a LIL for this setting.

THEOREM 2. If $\{X_n,\ n \geq 1\}$ is a sequence of exchangeable r.v.'s with $EX_1^2 < \infty$, then in (1.17) T is an extended random variable which can be defined by

$$T = \begin{cases} \infty & \text{on } \{E(X_1 \mid T\} > 0\} \\ \sqrt{E(X_1^2 \mid T)} & \text{on } \{E(X_1, \mid T) = 0\} \\ -\infty & \text{on } \{E(X_1 \mid T) < 0\} \end{cases} \tag{1.18}$$

REMARK. Traditionally, hypotheses of limit theorems for exchangeable random variables are phrased in terms of $\nu_F = E_F(X_1)$ and $\sigma_F^2 = E_F(X_1^2) - \nu_F^2$ which are random variables on the probability space (Φ, Σ, μ). It can be shown that $g(\omega) = P(X_1 \leq t \mid T)(\omega)$ is a measurable mapping of $(\Omega, \mathcal{A})$ into (Φ, Σ) and μ can be identified with the induced probability measure P_g

where T is any σ-field which make the exchangeable r.v.'s $\{X_n, n \geq 1\}$ conditionally i.i.d. (e.g., T could be the tail σ-field). Hence, T can be identified with T_F, a r.v. on (Φ, Σ, μ) defined by

$$T_F = \begin{cases} \infty & \text{on } \{F: \nu_F > 0\} \\ \sigma_F & \text{on } \{F: \nu_F = 0\}. \\ -\infty & \text{on } \{F: \sigma_F < 0\} \end{cases} \tag{1.19}$$

and the proof of Theorem 2 follows from the proof of Theorem 1. Note that $T = T_F \, og$ a.s. where o denotes the composition mapping.

PROOF OF THEOREM 2. Since $EX_1^2 < \infty$, ν_F, and σ_F^2 exist for μ - almost every $F \in \Phi$. From (1.12) if follows that

$$P_F\left[\limsup_{n\to\infty} \sum_{j=1}^{n} (X_j - \nu_F) \; / \; \sqrt{2n \, loglog \, n} = \sigma_F\right] = 1,$$

for μ - almost every $F \in \Phi$. The proof then follows by observing that

$$T_F = \limsup_{n\to\infty} \left\{ \sum_{j=1}^{n} (X_j - \nu_F) \; / \; \sqrt{2n \, loglog \, n} \; + \; \sqrt{n/(2 \, loglog \, n)} \;\; \nu_F \right\}$$

completing the proof. □

From the proof of Theorem 2, it is clear that the hypothesis $EX_1^2 < \infty$ can be replaced with $E_F X_1^2 < \infty$ μ -a.s.

ACKNOWLEDGEMENT. These authors are indebted to the referee for the constructive comments and very excited for his/her positive assessment. Also, example 2 in this paper is provided by the referee.

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