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# A SURVEY OF PARTIAL DIFFERENTIAL EQUATIONS WITH PIECEWISE CONTINUOUS ARGUMENTS

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**ABSTRACT.** Some work is described and new topics are posed on initial and boundary-value problems for partial differential equations whose arguments have intervals of constancy. These equations are of considerable theoretical and applied interest.

**KEY WORDS AND PHRASES.** Partial differential equations, piecewise constant delay, boundary value problem, initial value problem, abstract Cauchy problem, Hilbert space, loaded equation, wave equation.

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## 1. INTRODUCTION.

Functional differential equations (FDE) with delay provide a mathematical model for a physical or biological system in which the rate of change of the system depends upon its past history. Although the general theory and basic results for FDE have by now been thoroughly investigated, the literature devoted to this area of research continues to grow very rapidly. The number of interesting works is very large, so that our knowledge of FDE has been substantially enlarged in recent years. Naturally, new important problems and directions arise continually in this intensively developing field.

The article summarizes the results in the study and addresses the need for further investigation of generalized solutions to broad classes of FDE. The survey concentrates on differential equations with piecewise continuous arguments (EPCA), the exploration of which has been initiated in our papers a few years ago. These equations arise in an attempt to extend the theory of FDE with continuous arguments to differential equations with discontinuous arguments. This task is also of considerable applied interest since EPCA include, as particular cases, impulsive and loaded equations of control theory and are similar to those found in some biomedical models. A typical EPCA contains arguments that are constant on certain intervals. A solution is defined as a continuous, sectionally smooth function that satisfies the equation within these intervals. Continuity of a solution at a point joining any two consecutive intervals leads to recursion relations for the solution at such points. Hence, the solutions are determined by a finite set of initial data, rather than by an initial function as in the case of general FDE. Therefore, underlying each EPCA is a dynamical system governed by a difference equation of a discrete argument which describes its stability, oscillation, and periodic properties. It is not surprising then that recent work on EPCA has caused a new surge in the study of difference equations. Of significant interest is the exploration of partial differential equations (PDE) with piecewise continuous delays. Boundary and initial-value problems for some EPCA with partial derivatives were considered and the behavior of their solutions investigated. The results were also extended to equations with positive definite operators in Hilbert spaces. This topic is of great theoretical, computational, and applied value since it opens the possibility of approximating complicated problems of mathematical physics by simpler EPCA.

It is well known that profound and close links exist between functional and functional differential equations. Thus the study of the first often enables one to predict properties of differential equations of neutral type. On the other hand, some methods for the latter in the special case when the argument deviation vanishes at individual points have been used to investigate functional equations. Functional equations are directly related to difference equations of a discrete argument, and bordering on difference equations are impulsive FDE with impacts and switching and loaded equations (that is, those including values of the unknown solution for given constant values of the argument). The argument deviations of the EPCA considered in the paper vanish at countable sets of points, and it would be interesting to investigate the relationship between EPCA and functional equations. Another deserving direction of future research is the exploration of hybrid systems consisting of EPCA and functional equations. Furthermore, EPCA are intrinsically closer to difference rather than to differential equations. Equations with piecewise constant delay can be used to approximate differential equations that contain discrete delays. It would be useful to draw a detailed comparison of the qualitative and asymptotic properties of differential equations with continuous arguments and their EPCA approximations, which has been widely used for ordinary differential equations and their difference approximations. Since the arguments of an EPCA have intervals of constancy we must relinquish smoothness of the solutions, but we still retain their continuity. This enables us to derive a homogeneous difference equation for the values of a solution at the endpoints of the intervals of constancy and to employ it in the study of the original EPCA, thus revealing remarkable asymptotic, oscillatory, and periodic properties of this type of FDE. Of course, it is possible to further generalize the definition of a solution for an EPCA, by abandoning its continuity, and to include in the framework of EPCA the impulsive functional differential equations.

A typical EPCA is of the form

$$x'(t) = f(t, x(t), x(h(t))), \tag{1.1}$$

where the argument $h(t)$ has intervals of constancy. For example, in [1], equations with $h(t) = [t]$, $[t-n]$, $t - n[t]$ were investigated, where $n$ is a positive integer and $[\cdot]$ denotes the greatest-integer function. Note that $h(t)$ is discontinuous in these cases, and although the equation fits within the general paradigm of delay differential or functional differential equations, the delays are discontinuous functions. Also note that the equation is nonautonomous, since the delays vary with $t$. Moreover, as we have mentioned, the solutions are determined by a finite set of initial data, rather than by an initial function, as in the case of general FDE. In fact, EPCA have the structure of continuous dynamical systems within intervals of certain lengths. Continuity of a solution at a point joining any two consecutive intervals then implies recursion relations for the solution at such points. Therefore, EPCA represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations.

An equation in which $x'(t)$ is given by a function of $x$ evaluated at $t$ and at arguments $[t], \cdots, [t-N]$, where $N$ is a non-negative integer, may be called of *retarded* or *delay type*. If the arguments are $t$ and $[t+1], \cdots, [t+N]$, the equation is of *advanced type*. If both types of arguments appear in the equation, it may be called of *mixed type*. If the derivative of highest order appears at $t$ and at another point, the equation is generally said to be of *neutral type*. All types of EPCA share similar characteristics. First of all, it is natural to pose the initial-value problem for such equations not on an interval, but at a number of individual points. Secondly, for ordinary differential equations with a continuous vector field the solution exists to the right and left of the initial $t$-value. For retarded FDE, this is not necessarily the case [2].

Furthermore, it appears that advanced equations, in general, lose their margin of smoothness, and the method of successive integration shows that after several steps to the right from the initial interval the solution may even not exist. However, two-sided solutions do exist for all types of EPCA. Finally, the problems for EPCA studied so far are closely related to ordinary difference equations and indeed have stimulated new work on these.

It is important to note that EPCA provide the simplest examples of differential equations capable of displaying chaotic behavior. For instance, following Ladas [3], one can see that the unique solution of the initial-value problem

$$x'(t) = 3x([t]) - x^2([t]), \; x(0) = c_0 \tag{1.2ab}$$

where $[t]$ is the greatest-integer function, has the property that

$$x(n+1) = 4x(n) - x^2(n), \; n = 0, 1, \cdots \tag{1.3}$$

If we choose $c_0 = 4sin^2(\pi/9)$, then the unique solution of this difference equation is

$$x(n) = 4sin^2\left(2^n \frac{\pi}{9}\right), \tag{1.4}$$

which has period three. By the well-known result [4] which states that "period three implies chaos," the solution of the above differential equation exhibits chaos. Furthermore, the equation of Carvalho and Cooke

$$x'(t) = ax(t)(1 - x([t])) \tag{1.5}$$

is analagous to the famous logistic differential equation, but $t$ in one argument has been replaced by $[t]$. As a result, the equation has solutions that display complicated dynamics [5]. It seems likely that other simple nonlinear EPCA may display other interesting behavior.

The numerical approximation of differential equations can give rise to EPCA in a natural way, although it is unusual to take this point of view. For example, the simple Euler scheme for a differential equation $x'(t) = f(x(t))$ has the form $x_{n+1} - x_n = hf(x_n)$, where $x_n = x(nh)$ and $h$ is the step size. This is equivalent to the EPCA

$$x'(t) = f(x([t/h]h)). \tag{1.6}$$

Impulsive differential equations and loaded equations of control theory fit within the general paradigm of EPCA. Another potential application of EPCA is the stabilization of hybrid control systems with feedback delay. By a hybrid system we mean one with a continuous plant and with discrete (sampled) controller. Some of these systems may be described by EPCA [6].

Considerable work on EPCA has been done in recent years. In each of the areas – existence, asymptotic behavior, periodic and oscillating solutions, approximation, application to control theory, biomedical models, and problems of mathematical physics – there appears to be ample opportunity for extending the known results. A brief survey of ordinary differential equations with piecewise continuous arguments is given in [7].

## 2. BOUNDARY-VALUE PROBLEMS.

The first fundamental paper [8] in this direction appeared in 1991. It has been shown in [8] that these equations naturally arise in the process of approximating PDE by using piecewise constant arguments. Thus, for example, if in the equation

$$u_t = a^2 u_{xx} - bu, \tag{2.1}$$

which describes heat flow in a rod with both diffusion $a^2 u_{xx}$ along the rod and heat loss (or gain) across the lateral sides of the rod, the lateral heat change is measured at discrete times, then we get an equation with piecewise constant argument (EPCA)

$$u_t(x,t) = a^2 u_{xx}(x,t) - bu(x,nh), \quad t \in [nh,(n+1)h], \qquad n = 0,1,\cdots \tag{2.2}$$

where $h > 0$ is some constant. This equation can be written in the form

$$u_t(x,t) = a^2 u_{xx}(x,t) - bu(x,[t/h]h), \tag{2.3}$$

where $[\,\cdot\,]$ designates the greatest-integer function.

The diffusion-convection equation

$$u_t = a^2 u_{xx} - ru_x \tag{2.4}$$

describes, for instance, the concentration $u(x,t)$ of a pollutant carried along in a stream moving with velocity $r$. The term $a^2 u_{xx}$ is the diffusion contribution and $-ru_x$ is the convection component. If the convection part is measured at discrete times $nh$, the process results in the equation

$$u_t(x,t) = a^2 u_{xx}(x,t) - ru_x(x,[t/h]h). \tag{2.5}$$

These examples indicate at the considerable potential of EPCA as an analytical and computational tool in solving some complicated problems of mathematical physics. Therefore, it is important to investigate boundary-value problems (BVP) and initial-value problems (IVP) for EPCA in partial derivatives, and explore the influence of certain discontinuous delays on the behavior of solutions to some typical problems of mathematical physics.

The topic of [8] is the BVP consisting of the equation

$$\frac{\partial u(x,t)}{\partial t} + P\left(\frac{\partial}{\partial x}\right) u(x,t) = Q\left(\frac{\partial}{\partial x}\right) u\left(x,\left[\frac{t}{h}\right] h\right), \tag{2.6}$$

where $P$ and $Q$ are polynomials of the highest degree $m$ with coefficients that may depend only on $x$, the boundary conditions

$$L_j u = \sum_{k=1}^{m} (M_{jk}u^{(k-1)}(0) + N_{jk}u^{(k-1)}(t)) = 0, \tag{2.7}$$

$$(M_{jk} \text{ and } N_{jk} \text{ are constants, } j = 1, \cdots, m)$$

and the initial condition

$$u(x,0) = u_0(x). \tag{2.8}$$

where $(x,t) \in [0,1] \times [0,\infty)$, and $h = \text{const.} > 0$. Conditions (2.7) will be written briefly as

$$Lu = 0. \tag{2.9}$$

An important result has been established that BVP (2.6)-(2.8) has a solution in $[0,1] \times [nh,(n+1)h]$, if the following hypotheses hold true:

(i) The boundary-value problem

$$P\left(\frac{d}{dx}\right)X - \lambda X = 0, \qquad LX = 0 \tag{2.10}$$

is self-adjoint, all its eigenvalues $\lambda_j$ are positive.

(ii) For each $\lambda_j$, the roots of the equation $P(z) - \lambda_j = 0$ have non-positive real parts.

(iii) The initial function $u_0(x) \in C^m[0,1]$ satisfies (2.7).

The solution $u_n(x,t)$ of BVP (2.6)-(2.8) on the interval $nh \le t < (n+1)h$ is represented in the form of a Fourier series

$$u_n(x,t) = \sum_{j=1}^{\infty} X_j(x) T_{nj}(t), \tag{2.11}$$

where $X_j(x)$ are the eigenvalues of the operator $P$. The functions $T_{nj}(t)$ are solutions of ordinary EPCA that arise after separation of variables.

For instance, in $[0,1] \times [nh,(n+1)h]$, the solution $u_n(x,t)$ of Eq. (2.3) with boundary conditions $u_n(x,nh) = u_n(x)$ is sought in form (2.11). Separation of variables produces

$$X_j(x) = \sqrt{2}sin(\pi jx), \qquad T'_{nj}(t) + a^2\pi^2 j^2 T_{nj}(t) = -bT_{nj}(nh), \tag{2.12}$$

whence

$$T_{nj}(t) = C_{nj}e^{-a^2\pi^2 j^2(t-nh)} - \frac{b}{a^2\pi^2 j^2}T_{nj}(nh). \tag{2.13}$$

We put $t = nh$ in this equation to obtain

$$C_{nj} = \left(1 + \frac{b}{a^2\pi^2 j^2}\right)T_{nj}(nh),$$

that is,

$$T_{nj}(t) = E_j(t-nh)T_{nj}(nh),$$

where

$$E_j(t) = e^{-a^2\pi^2 j^2 t} - \left(1 - e^{-a^2\pi^2 j^2 t}\right)\frac{b}{a^2\pi^2 j^2}. \tag{2.14}$$

At $t = (n+1)h$ we have

$$T_{nj}((n+1)h) = E_j(h)T_{nj}(nh)$$

and since

$$T_{nj}((n+1)h) = T_{n+1,j}((n+1)h),$$

then

$$T_{n+1,j}((n+1)h) = E_j(h)T_{nj}(nh)$$

and

$$T_{nj}(nh) = E_j^n(h)T_{0j}(0).$$

Therefore,

$$T_{nj}(t) = E_j(t-nh)E_j^n(h)T_{0j}(0) \tag{2.15}$$

and

$$u_n(x,t) = \sum_{j=1}^{\infty} \sqrt{2}E_j^n(h)T_{0j}(0)E_j(t-nh)sin(\pi jx). \tag{2.16}$$

Putting $t = 0, n = 0$ gives

$$u_0(x) = \sum_{j=1}^{\infty} T_{0j}(0)\sqrt{2}sin(\pi jx)dx \tag{2.17}$$

where

$$T_{0j}(0) = \sqrt{2}\int_0^1 u_0(x)sin(\pi jx)dx.$$

If $|E_j(h)| < 1$, then solution (2.16) decays exponentially as $t \to \infty$, uniformly with respect to $x$. From (2.14) it follows that this is true if

$$-a^2\pi^2 < b < a^2\pi^2 \frac{e^{a^2\pi^2h}+1}{e^{a^2\pi^2h}-1}.$$

Furthermore, from the equations

$$T_{nj}(nh) = E_j^n(h)T_{0j}(0), \qquad T_{nj}((n+1)h) = E_j^{n+1}(h)T_{0j}(0)$$

we see that $T_{nj}(nh)T_{nj}((n+1)h) < 0$ if $E_j(h) < 0$. The latter inequality holds true if

$$b > \frac{a^2\pi^2}{e^{a^2\pi^2h}-1}. \tag{2.18}$$

Hence, under condition (2.18) each function $T_{nj}(t)$ $(j = 1,2,\cdots)$ has a zero in the interval $[nh,(n+1)h]$, in sharp contrast to the functions $T_j(t)$ in the Fourier expansion for the solution of the equation $u_t = a^2a_{xx} - bu$ without time delay. Moreover, the inequality $E_j(h) < 0$ takes place for sufficiently large $j$ and any $b > 0$. Therefore, for $b > 0$ and sufficiently large $j$, the functions $T_{nj}(t)$ are oscillatory.

Eq. (2.5) on $nh \le t < (n+1)h$ becomes

$$\frac{\partial u_n(x,t)}{\partial t} = a^2\frac{\partial^2 u_n(x,t)}{\partial x^2} - ru_n'(x),$$

and we differentiate the latter with respect to $t$ to obtain the equation

$$\frac{\partial y_n}{\partial t} = a^2\frac{\partial^2 y_n}{\partial x^2}, \qquad y_n = \frac{\partial u_n}{\partial t},$$

whose solution is sought in form (2.11). Separation of variables leads to the equations

$$X''(x) + \lambda X(x) = 0, \quad T_n'(t) + a^2\lambda T_n(t) = 0, \tag{2.19}$$

and the boundary conditions $u_n(0,t) = u_n(1,t) = 0$ give $\lambda_j = j^2\pi^2$ and

$$y_n(x,t) = \sum_{j=1}^{\infty} \sqrt{2}\, T_{nj}(nh)e^{-a^2\pi^2 j^2(t-nh)} sin(\pi j x). \tag{2.20}$$

Since

$$y_n(x,nh) = a^2u_n''(x) - ru_n'(x), \qquad u_n(x) = u_n(x,nh),$$

then

$$a^2u_n''(x) - ru_n'(x) = \sum_{j=1}^{\infty} \sqrt{2}\, T_{nj}(nh) sin(\pi j x)$$

and

$$T_{nj}(nh) = -a^2\pi^2 j^2\sqrt{2}\int_0^1 u_n(x) sin(\pi j x)dx + r\pi j\sqrt{2}\int_0^1 u_n(x) cos(\pi j x)dx.$$

Finally

$$u_n(x,t) = u_n(x) + \sum_{j=1}^{\infty} \frac{\sqrt{2}T_{nj}(nh)(1 - e^{-a^2\pi^2 j^2(t-nh)}) sin(\pi j x)}{a^2\pi^2 j^2}. \tag{2.21}$$

Given the initial function $u(x,0) = u_0(x)$, we can find the coefficients $T_{0j}(0)$ and the solution $u_0(x,t)$ on $0 \le t \le h$. Since $u_0(x,h) = u_1(x)$, we can calculate the coefficients $T_{1j}(h)$ and the solution $u_1(x,t)$ on $h \le t \le 2h$. By the method of steps the solution can be extended to any interval $[nh,(n+1)h]$.

The equation

$$iq\frac{\partial u(x,t)}{\partial t} = -\frac{q^2}{2m_0}\frac{\partial^2 u(x,t)}{\partial x^2} + V(x)\, u\left(x,\left[\frac{t}{h}\right]h\right) \tag{2.22}$$

is a piecewise constant analogue of the one-dimensional Schrödinger equation

$$iq\psi_t(x,t) = \frac{-q^2}{2m_0}\psi_{xx}(x,t) + V(x)\psi(x,t). \tag{2.23}$$

If $u(x,t)$ satisfies conditions (2.4) and (2.5), with $m = 2$, then separation of variables produces a formal solution

$$u_n(x,t) = \sum_{j=1}^{\infty} C_{nj} exp[-\lambda_j(t-nh)/q]\, X_j(x) + P^{-1}Qu_n(x), \tag{2.24}$$

for $nh \le t \le (n+1)h$. Here, $X_j(x)$ are the eigenfunctions of the operator $q^2(d^2/dx^2)/2m_0$, and $P^{-1}Qu_n(x)$ is the solution $v_n(x)$ of the equation

$$q^2v_n''(x) = 2m_0V(x)u_n(x)$$

that satisfies (2.7).

The Fourier method was also used to find weak solutions of the boundary-value problem (2.6)-(2.8) and it is easily generalized to similar problems in Hilbert space. First, we remind a few well-known definitions. Let $H$ be a Hilbert space and let $P$ be a linear operator in $H$ (additive and homogeneous but, possibly, unbounded) whose domain $\mathfrak{D}(P)$ is dense in $H$,, that is $\overline{\mathfrak{D}(P)} = H$. The operator $P$ is called symmetric if $(Pu,v) = (u,Pv)$, for any $u,v \in \mathfrak{D}(P)$. If $P$ is symmetric, then $(Pu, v)$ is a symmetric bilinear functional and $(Pu, u)$ is a quadratic form. A symmetric operator $P$ is called positive if $(Pu, u) \geq 0$ and $(Pu, u) = 0$ if and only if $u = 0$. A symmetric operator $P$ is called positive definite if there exists a constant $\gamma^2 > 0$ such that $(Pu, u) \geq \gamma^2 \| u \|^2$. With every positive operator $P$ a certain Hilbert space $H_P$ can be associated, which is called the energy space of $P$. It is the completion of $\mathfrak{D}(P)$, with the inner product $(u, v)_P = (Pu, v)$; $u, v \in \mathfrak{D}(P)$. This product induces a new norm $\| u \|_P = (Pu, u)^{1/2}, u \in \mathfrak{D}(P)$, and if $P$ is positive definite, then $\| u \| \leq \gamma^{-1} \| u \|_P$. Since $\mathfrak{D}(P)$ is dense in $H$, it follows by using the latter inequality that the energy space $H_P$ of a positive definite operator $P$ is dense in the original space $H$.

Assuming $P$ is positive definite, we may consider the solution $u(x, t)$ of (2.6)-(2.8) for a fixed $t$ as an element of $H_P$. If $\mathfrak{D}(Q) \subset H$, then $Qu(x, [t/h]h)$ may be treated as an abstract function $Qu([t/h]h)$ with the values in $H$. Therefore, the given $BVP$ is reduced to the abstract Cauchy problem

$$\frac{du}{dt} + Pu = Qu\left(\left[\frac{t}{h}\right]h\right), \qquad t > 0, u\,|_{\,t=0} = u_0 \in H. \tag{2.25}$$

If (2.25) has a solution, we multiply each term by an arbitrary function $g(t) \in H_P$ in the sense of inner product in $H$ and get on the interval $nh \leq t < (n+1)h$ the equation

$$\left(\frac{du}{dt}, g\right) + (u, g)_P = (Qu_n, g), \tag{2.26}$$

where $u_n = u(nh)$. Conversely, if $u \in C^1((nh, (n+1)h); \mathfrak{D}(P))$ for all integers $n \geq 0$ and satisfies (2.26), then it also satisfies (2.25). Indeed, if $u \in \mathfrak{D}(P)$, then $(u, g)_P = (Pu, g)$, and (2.26) can be written as

$$\left(\frac{du}{dt} + Pu - Qu_n, g\right) = 0, \qquad nh \leq t < (n+1)h.$$

Since $H_P$ is dense in $H$, then $u(t)$ is a solution of (2.25).

**DEFINITION.** An abstract function $u(t): [0, \infty) \to H$ is called a *weak solution* of problem (2.25) if it satisfies the conditions:

(i) $u(t)$ is continuous for $t \geq 0$ and strongly continuously differentiable for $t > 0$, with the possible exception of the points $t = nh$ where one-sided derivatives exist.

(ii) $u(t)$ is continuous for $t > 0$ as an abstract function with the values in $H_P$ and satisfies (2.26) on each interval $nh \leq t < (n+1)h$, for any function $g(t): [0, \infty) \to H_P$.

(iii) $u(t)$ satisfies the initial condition (2.25), that is,

$$\lim_{t \to 0} \| u(t) - u_0 \|_H = 0.$$

A weak solution $u(t)$ is also an ordinary solution if $u(t) \in \mathfrak{D}(P)$, for any $t > 0$, and $u(x, t) \to u_0(x)$ as $t \to 0$ not only in the norm of $H$ but uniformly as well. It is said that a symmetric operator $P$ has a discrete spectrum if it has an infinite sequence $\{\lambda_j\}$ of eigenvalues with a single limit point at infinity and a sequence $\{X_j\}$ of eigenfunctions which is complete in $H$. Suppose the operator $P$ in (2.26) is positive definite and has a discrete spectrum and assume the existence of a solution $u(t) = u(x, t)$ to (2.26) with the condition $u(0) = u_0$. On the interval $nh \leq t < (n+1)h$ this solution can be expanded into series (2.11), where $T_j(t) = (u(t), X_j)$. To find the coefficients $T_j(t)$, we put $g(t) = X_k$ in (2.26) and since $X_k$ does not depend on $t$, then

$$\left(\frac{du(t)}{dt}, X_k\right) = \frac{d}{dt}(u(t), X_k) = T_k'(t),$$

$$(u, X_k)_P = (Pu, X_k) = (u, PX_k) = \lambda_k(u, X_k) = \lambda_k T_k(t),$$

which leads to the equation

$$T_{n_j}'(t) + \lambda_j T_{n_j}(t) = (Qu_n, X_j)$$

By selecting a proper space $H$, a weak solution corresponding to conditions (2.7) can be constructed. A theorem has been stated in [8] that if $P$ and $Q$ are linear operators in a Hilbert space and $P$ is positive definite with a discrete spectrum, then there exists a unique weak solution of problem (2.25).

## 3. INITIAL-VALUE PROBLEMS.

The topic has been explored by Wiener and Debnath in [9]. Eq. (2.6) with constant coefficients and initial condition (2.8) has been considered in the domain

$$(x,t) \in \Omega = (-\infty, \infty) \times [0, \infty).$$

Let $u_n(x,t)$ be the solution of the given problem on $nh \leq t < (n+1)h$, then

$$\frac{\partial u_n(x,t)}{\partial t} + Pu_n(x,t) = Qu_n(x), \tag{3.1}$$

where

$$u_n(x) = u_n(x, nh). \tag{3.2}$$

Write

$$u_n(x,t) = w_n(x,t) + v_n(x),$$

which gives the equation

$$\frac{\partial w_n}{\partial t} + Pw_n + Pv_n(x) = Qu_n(x),$$

and require that

$$\frac{\partial w_n}{\partial t} + Pw_n = 0, \tag{3.3}$$

$$Pv_n(x) = Qu_n(x). \tag{3.4}$$

If $v_n(x)$ is a solution of ODE (3.4), then at $t = nh$ we have

$$w_n(x, nh) = u_n(x) - v_n(x), \tag{3.5}$$

and it remains to consider (3.3) with initial condition (3.5). It is well known that the solution $E(x,t)$ of the problem

$$\frac{\partial w}{\partial t} + Pw = 0, \qquad w(x,0) = w_0(x), \tag{3.6}$$

with $w_0(x) = \delta(x)$, where $\delta(x)$ is the Dirac delta functional, is called its *fundamental solution*. The solution of IVP (3.6) is given by the convolution

$$w(x,t) = E(x,t) * w_0(x). \tag{3.7}$$

Hence, the solution of problem (3.3), (3.5) can be written as

$$w_n(x,t) = E(x,t-nh)*w_n(x,nh), \tag{3.8}$$

and the solution of (3.1), (3.2) is

$$u_n(x,t) = E(x,t-nh)*(u_n(x) - v_n(x)) + v_n(x), \qquad (nh \le t < (n+1)h). \tag{3.9}$$

Continuity of the solution at $t = (n+1)h$ implies

$$u_n(x,(n+1)h) = u_{n+1}(x,(n+1)h) = u_{n+1}(x),$$

that is,

$$u_{n+1}(x) = E(x,h)*(u_n(x) - v_n(x)) + v_n(x). \tag{3.10}$$

Formulas (3.9) and (3.10) successively determine the solution of IVP (2.6), (2.8) on each interval $nh \le t \le (n+1)h$. Indeed, from $Pv_0(x) = Qu_0(x)$ we find $v_0(x)$ and substitute both $u_0(x)$ and $v_0(x)$ in (3.9) and (3.10) to obtain $u_0(x,t)$ and $u_1(x)$. Then we use $u_1(x)$ in (3.4) to find $v_1(x)$ and substitute $u_1(x)$ and $v_1(x)$ in (3.9) and (3.10), which yields $u_1(x,t)$ and $u_2(x)$. Continuing this procedure leads to $u_n(x,t)$, the solution of (2.6) on $[nh,(n+1)h]$. The solution $v_n(x)$ of (3.4) is defined to within an arbitrary polynomial $q(x)$ of degree $< m$. Since $q(x)$ is a solution of (3.6) with the initial condition $w(x,0) = q(x)$, then $q(x) = E(x,t)*q(x)$, and $q(x)$ cancels in the formulas (3.9) and (3.10). This proves that if (3.6) with $w(x,0) = u_0(x)$ has a unique solution on $t \in (0,\infty)$, then there exists a unique solution of IVP (2.6), (2.8) on $(0,\infty)$ and it is given by (3.9) for each interval $nh \le t \le (n+1)h$. Thus, there exist unique solutions of (2.3) and (2.5), with $u(x,0) = u_0(x)$; in the class of functions that grow to infinity slower that $exp(x^2)$ as $|x| \to \infty$. For (2.3) and (2.5) we have

$$v_n(x) = a^{-2}b\int_0^x (x-s)u_n(s)ds \text{ and } v_n(x) = a^{-2}r\int_0^x u_n(s)ds,$$

respectively, and $E(x,t) = exp(-x^2/4a^2t)/2a\sqrt{\pi t}$.

Formula (3.9) for the solution of the problem

$$u_t(x,t) = a^2u_{xx}(x,t) - bu_{xx}\left(x,\left[\frac{t}{h}\right]h\right), u(x,0) = u_0(x)$$

on $nh \le t < (n+1)h$ becomes

$$u_n(x,t) = \left(1 - \frac{b}{a^2}\right)E(x,t-nh)*u_n(x) + \frac{b}{a^2}\,u_n(x),$$

where $E(x,t)$ is the same as in (2.3) and (2.5).

The above method may also be used to solve IVP for PDE of any order in $t$ with piecewise constant delay or systems of such equations. In the latter case, $P$ and $Q$ in (2.6) are square matrices of linear differential operators and $u(x,t)$ is a vector function. Thus, the solution $u_n(x,t)$ of the problem

$$u_{tt}(x,t) = a^2u_{xx}(x,t) - bu_{xx}(x,[t]),$$

$$u(x,0) = f_0(x), u_t(x,0) = g_0(x)$$

on $n \le t < n+1$ is sought in the form $u_n(x,t) = w_n(x,t) + v_n(x)$ whence $a^2v_n''(x) - bu_n''(x,n) = 0$ and $\partial^2 w_n/\partial t^2 = a^2\partial^2 w_n/\partial x^2$. Setting $u(x,n) = f_n(x), u_t(x,n) = g_n(x)$ gives

$$v_n(x) = a^{-2}bf_n(x), \quad w(x,n) = (1-a^{-2}b)f_n(x), \quad w_t(x,n) = g_n(x),$$

and

$$u_n(x,t) = \frac{b}{a^2} f_n(x) + \left(1 - \frac{b}{a^2}\right)\frac{f_n(x-a(t-n)) + f_n(x+a(t-n))}{2} + \frac{1}{2a}\int_{x-a(t-n)}^{x+a(t-n)} g_n(s)ds.$$

Putting $t = n+1$ produces the recursion relations

$$f_{n+1}(x) = \frac{b}{a^2} f_n(x) + \left(1 - \frac{b}{a^2}\right) \frac{f_n(x-a) + f_n(x+a)}{2} + \frac{1}{2a}\int_{x-a}^{x+a} g_n(s)ds,$$

$$g_{n+1}(x) = \left(1 - \frac{b}{a^2}\right)\frac{af_n'(x+a) - af_n'(x-a)}{2} + \frac{1}{2}\left(g_n(x+a) + g_n(x-a)\right).$$

Loaded partial differential equations have properties similar to those of equations with piecewise constant delay. The IVP for the following class of loaded equations

$$\frac{\partial u(x,t)}{\partial t} = P\left(\frac{\partial}{\partial x}\right)u(x,t) + \sum_{j=1}^{q} Q_j\left(\frac{\partial}{\partial x}\right)u(x,t_j), \tag{3.11}$$

$$u(x,0) = u_0(x)$$

was considered in [9] and [10], where $(x,t) \in \mathbb{R}^n \times [0,T]$, the $t_j \in (0,T]$ are given, $P(s)$ and $Q(s)$ are polynomials in $s = (s_1, \cdots, s_n)$, and $\Sigma \mid Q_j(s) \mid \neq 0$. Eq. (3.11) arises in solving certain inverse problems for systems with elements concentrated at specific moments of time. The Fourier transform $U(s,t)$ of $u(x,t)$ satisfies the equation

$$U_t(s,t) = P(is)U(s,t) + \sum_{j=1}^{q} Q_j(is)U(s,t_j),$$

whence,

$$U(s,t) = U_0(s)e^{P(is)t} + k(P(is),t)\sum_{j=1}^{q} Q_j(is)U(s,t_j), \tag{3.12}$$

where $U_0(s)$ is the Fourier transform of $u_0(x)$ and

$$k(P(is),t) = \int_0^t exp\{P(s)y\}dy.$$

Denote

$$A_j = U_0(s)e^{P(is)t_j}, \; k_j = k(P(is),t_j), \; B = \sum_{j=1}^{q} Q_j(is)U(s,t_j), \tag{3.13}$$

then multiply by $Q_j(is)$ each of the equations

$$U(s,t_j) = A_j + k_jB, \quad j = 1, \cdots, q$$

and add them. Hence,

$$B = \sum_{j=1}^{q} A_jQ_j(is) + B\sum_{j=1}^{q} k_jQ_j(is)$$

or

$$\left(1 - \sum_{j=1}^{q} k_jQ_j(is)\right)B = \sum_{j=1}^{q} A_jQ_j(is). \tag{3.14}$$

The equation

$$\Delta(s) \equiv 1 - \sum_{j=1}^{q} Q_j(is)k(P(is), t_j) = 0 \tag{3.15}$$

is called the *characteristic equation* for (3.11) and its solution set $Z$ is called the *characteristic variety* of (3.11). It is said [10] that (3.11) is absolutely nondegenerate if $Z = \emptyset$, nondegenerate of type $a$ if

$$a = inf\,|\,Im\ s\,| < \infty, \qquad s \in Z \neq C^n,$$

and degenerate if $Z = C^n$. The case $Z = \emptyset$ implies $\Delta(s) =$ constant, since $\Delta(s)$ is meromorphic, and a meromorphic function that is not constant assumes every complex value with at most two exceptions. The equation $\Delta(s) = C$ can be written as

$$P(is) + \sum_{j=1}^{q} Q_j(is) - \sum_{j=1}^{q} Q_j(is)exp(P(is)t_j) = CP(is)$$

and is possible for $q > 1$ only if $P(s) =$ constant, otherwise $exp(P(is)t_j)$ would grow faster than any polynomial, which breaks the latter equality. For $q = 1$ we have

$$\Delta(s) = \frac{P(is) + Q_1(is) - Q_1(is)\ e^{P(is)t_1}}{P(is)},$$

and in this case $Z = \emptyset$ is equivalent to $P(is) + Q_1(is) \equiv 0$. On the other hand, $\Delta(s) \equiv 0$ is equivalent to

$$P(is) + \sum_{j=1}^{q} Q_j(is) - \sum_{j=1}^{q} Q_j(is)e^{P(is)t_j} \equiv 0,$$

which implies $P(s) =$ constant. This establishes the following proposition which was stated in [10] without proof, namely (3.11) is absolutely nondegenerate if and only if either of the following conditions holds true:

(i) $P(s) \equiv C_1, \quad \sum_{j=1}^{q} Q_j(s)k(C_1, t_j) \equiv C_2 \neq 1;$

or

(ii) $q = 1, \qquad P(s) + Q_1(s) \equiv 0.$

Eq. (3.11) is degenerate if and only if

$$P(s) \equiv C_1, \ \sum_{j=1}^{q} Q_j(s)k(C_1, t_j) \equiv 1.$$

Substituting $B$ from (3.14) in (3.12) leads to the proof that the uniqueness classes for the solution of the Cauchy problem for an absolutely nondegenerate equation (3.11) are the same as those for the equation (without "loads") $u_t(x,t) = Pu(x,t)$. The homogeneous degenerate IVP (3.11) ($u_0(x) = 0$) has nontrivial solutions, with compact support. Suppose that (3.11) is of finite type $a(0 < a < \infty)$ and that $u(x,t)$ is a solution of (3.11) with $u_0(x) \equiv 0$. If

$$|\,u(x,t)\,| \leq Ce^{\alpha|x|}, \qquad x \in \mathbb{R}^n, \qquad t \in [0,t], \tag{3.16}$$

and $\alpha < a$, then $u(x,t) \equiv 0$. For any $\alpha > a$ there exists a solution $u(x,t) \neq 0$ of (3.11) with $u_0(x) \equiv 0$ satisfying (3.16). Integral transformations have also been used in the study of EPCA.

Consider the nonlinear initial-value problem

$$\frac{\partial u}{\partial t} = A(D)u(x,t) + f(t, u(x,[t])), \qquad (3.17)$$
$$u(x,0) = u_0(x),$$

where $u(x,t)$ and $u_0(x)$ are $m$-vectors, $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$,

$$A(D) = \sum_{|\alpha| \leq r} A_\alpha D^\alpha,$$

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N), \; |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N,$$

$$D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}, D_k = i\partial/\partial x_k (k = 1, 2, \cdots, N),$$

the coefficients $A_\alpha$ are given constant matrices of order $m \times m$, and the $m$-vector $f \in C^1([n, n+1) \times \mathcal{L}^2(\mathbb{R}^N), \mathcal{L}^2(\mathbb{R}^N))$ $n = 0, 1, 2, \cdots$. The number $r$ is called the *order* of the system. It is assumed that $u_0 \in \mathcal{L}^2(\mathbb{R}^N)$, and the solutions sought are such that $u(x,t) \in \mathcal{L}^2(\mathbb{R}^N)$, for every $t \geq 0$. Let $\mu_1(s), \mu_2(s), \cdots, \mu_m(s)$ be the eigenvalues of the matrix $A(s)$. The system

$$\frac{\partial u}{\partial t} = A(D)u \qquad (3.18)$$

is said to be parabolic by Shilov if

$$Re\; \mu_j(s) \leq -c|s|^h + b, \qquad j = 1, \cdots, m$$

where $h > 0, c > 0$, and $b$ are constants. For a fixed $t$ we may consider the solution $u(x,t)$ as an element of $\mathcal{L}^2(\mathbb{R}^N)$, and $f(t, u(x,[t]))$ may be treated as an abstract function $f(t, u([t]))$ with the values in $\mathcal{L}^2$. Therefore, IVP (3.17) is reduced to the abstract Cauchy problem

$$\frac{du}{dt} = Au + f(t, u([t])), \qquad u|_{t=0} = u_0 \in \mathcal{L}^2. \qquad (3.19)$$

Applying to (3.18), with the initial condition $u(x,0) = u_0(x)$, the Fourier transformation $\mathfrak{F}$ in $x$ produces the system of ordinary differential equations

$$U_t(s,t) = A(s)U(s,t), \qquad (3.20)$$

with the initial condition $U(s,0) = U_0(s)$, where $U(s,t) = \mathfrak{F}(u(x,t)), U_0(s) = \mathfrak{F}(u_0(x))$, and $A(s)$ is a matrix with polynomial entries depending on $s = (s_1, s_2, \cdots, s_N)$. The solution of (3.20) is given by the formula

$$U(s,t) = e^{tA(s)}U_0(s).$$

Parabolicity of (3.18) by Shilov implies that the semigroup $T(t)$ of operators of multiplication by $e^{tA(s)}$, for $t > 0$, is an infinitely smooth semigroup of operators bounded in $\mathcal{L}^2(\mathbb{R}^N)$. Together with the requirement $h = r$, this ensures that the Cauchy problem for (3.18) is uniformly correct in $\mathcal{L}^2(\mathbb{R}^N)$ and all its solutions are infinitely smooth functions of $t$, for $t > 0$. Since $f$ is continuously differentiable, problem (3.17) has a unique solution on [0,1)

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u_0)ds.$$

Denoting $u_1 = u(1)$, we can find the solution

$$u(t) = T(t-1)u_1 + \int_1^t T(t-s)f(s,u_1)ds$$

of (3.17) on [1,2) and continue this procedure successively. If

$$f(t,u([t])) = Bu([t]),$$

where $B$ is a constant matrix, the solution of (3.17) for $t \in [0,\infty)$ is given by

$$u(t) = \left(T(t-[t]) + \int_{[t]}^t T(t-s)Bds\right) \prod_{k=[t]}^{1} \left(T(1) + \int_{k-1}^{k} T(k-s)Bds\right)u_0.$$

This proves that problem (3.17) has a unique solution on $\mathbb{R}^N \times [0,\infty)$ if system (3.18) is parabolic by Shilov, the index of parabolicity $h$ coincides with its order $r$, and $f \in C^1([n,n+1]) \times \mathcal{L}^2(\mathbb{R}^N)$, $\mathcal{L}^2(\mathbb{R}^N))$, $n = 0,1,2,\cdots$.

## 4. WAVE EQUATIONS WITH DISCONTINUOUS TIME DELAY.

The influence of terms with piecewise constant time on the behavior of the solutions, especially their oscillatory properties, of the wave equation was initiated in 1991 by Wiener and Debnath ([11], [12]).

First, we shall discuss separation of variables in systems of PDE. Consider the BVP consisting of the equation

$$U_t(x,t) = AU_{xx}(x,t) + BU_{xx}(x,[t]), \tag{4.1}$$

the boundary conditions

$$U(0,t) = U(1,t) = 0, \tag{4.2}$$

and the initial condition

$$U(x,0) = U_0(x). \tag{4.3}$$

Here, $U(x,t)$ and $U_0(x)$ are real $m \times m$ matrices, $A$ and $B$ are real constant $m \times m$ matrices and $[\cdot]$ denotes the greatest-integer function. Looking for a solution in the form

$$U(x,t) = T(t)X(x) \tag{4.4}$$

gives

$$T'(t)X(x) = AT(t)X''(x) + BT([t])X''(x),$$

whence

$$(AT(t) + BT([t]))^{-1}T'(t) = X''(x)X^{-1}(x) = -P^2,$$

which generates the BVP

$$X''(x) + P^2X(x) = 0, \tag{4.5}$$
$$X(0) = X(1) = 0$$

and the equation with piecewise constant argument

$$T'(t) = -AT(t)P^2 - BT([t])P^2. \tag{4.6}$$

The general solution of (4.5) is

$$X(x) = cos(xP)C_1 + sin(xP)C_2,$$

where

$$cos(xP) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} P^{2n}}{(2n)!}, \quad sin(xP) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} P^{2n+1}}{(2n+1)!}$$

and $C_1, C_2$ are arbitrary constant matrices. From $X(0) = 0$ we conclude that $C_1 = 0$, and the condition $X(1) = 0$ enables us to choose $sin P = 0$ (although this is not the necessary consequence of the equation $(sin P)C_2 = 0$). This can be written $e^{iP} - e^{-iP} = 0, e^{2iP} = I$. Assuming that all eigenvalues $p_1, p_2, \cdots, p_m$ of $P$ are distinct and $S^{-1}PS = \mathfrak{D} = diag(p_1, p_2, \cdots, p_m)$, we have $exp(2iS\mathfrak{D}S^{-1}) = I, Se^{2i\mathfrak{D}}S^{-1} = I$, and $e^{2i\mathfrak{D}} = I$. Therefore, $\mathfrak{D} = diag(\pi j_1, \pi j_2, \cdots, \pi j_m)$, where the $j_k$ are integers, and $P = S\mathfrak{D}S^{-1}$,

$$P^2 = S\mathfrak{D}^2 S^{-1} = S\ diag(\pi^2 j_1^2, \pi^2 j_2^2, \cdots, \pi^2 j_m^2) S^{-1},$$

$sin(xP) = S\ sin(x\mathfrak{D})S^{-1} = S\ diag(sin \pi j_1 x, \cdots, sin \pi j_m x) S^{-1}$. Furthermore, we can put

$$P_j = diag(\pi(m(j-1)+1), \cdots, \pi m j), \quad (j = 1, 2, \cdots) \tag{4.7}$$

in (4.5) and obtain the following result:

There exists an infinite sequence of matrix eigenfunctions for BVP (4.5)

$$X_j(x) = \sqrt{2}\ diag(sin \pi(m(j-1)+1)x, \cdots, sin \pi m j x), \quad (j = 1, 2, \cdots) \tag{4.8}$$

which is complete and orthonormal in the space $\mathfrak{L}^2[0,1]$ of $m \times m$ matrices, that is,

$$\int_0^1 X_j(x) X_k(x) ds = \begin{cases} 0, & j \neq k \\ I, & j = k \end{cases}$$

where $I$ is the identity matrix.

Let $E(t)$ be the solution of the problem

$$T'(t) = -AT(t)P^2, \qquad T(0) = I \tag{4.9}$$

and let

$$M(t) = E(t) + (E(t) - I)A^{-1}B. \tag{4.10}$$

If the matrix $A$ is nonsingular, then (4.6) with the initial condition $T(0) = C_0$ has on $[0, \infty)$ a unique solution

$$T(t) = M(t - [t]) M^{[t]}(1) C_0. \tag{4.11}$$

If $\| M(1) \| < 1$, then $\| T(t) \|$ exponentially tends to zero as $t \to +\infty$.

For the scalar parabolic equation

$$u_t(x,t) = a^2 u_{xx}(x,t) + b u_{xx}(x,[t])$$

we have $m = 1$ and $P_j = \pi j$, according to (4.7). For (4.9) with $A = a^2$ and $P = P_j$, we have $E_j(t) = exp(-a^2\pi^2 j^2 t)$ and

$$M_j(t) = e^{-a^2\pi^2 j^2 t} - \frac{b}{a^2}(1 - e^{-a^2\pi^2 j^2 t}).$$

Hence, the inequality $| M_j(1) | < 1$ is equivalent to

$$-1 < e^{-a^2\pi^2 j^2} - \frac{b}{a^2}(1 - e^{-a^2\pi^2 j^2}) < 1,$$

whence

$$-a^2 < b < a^2\ \frac{1+e^{-a^2\pi^2 j^2}}{1-e^{-a^2\pi^2 j^2}}.$$

Since the function $(1+e^{-t})/(1-e^{-t})$ is decreasing, all functions $T_j(t)$ exponentially tend to zero as $t \to \infty$ if and only if

$$-a^2 < b \leq a^2. \tag{4.12}$$

If $b < -a^2$, then all $T_j(t)$ monotonically tend to infinity as $t \to \infty$; and if

$$b > a^2\ \frac{1+e^{-a^2\pi^2}}{1-e^{-a^2\pi^2}},$$

then all $T_j(t)$ are unbounded and oscillatory. For any $b > a^2$, there exists a positive integer $j_0$ such that the $T_j(t)$ are unbounded and oscillatory, for $j > j_0$. Indeed, letting $b = a^2 + \epsilon$ and solving the inequality

$$a^2 + \epsilon > a^2\ \frac{1+e^{-a^2\pi^2 j^2}}{1-e^{-a^2\pi^2 j^2}}.$$

gives

$$e^{-a^2\pi^2 j^2} < \frac{\epsilon}{2a^2+\epsilon},$$

which holds for any positive $\epsilon$ and sufficiently large $j$ and implies $M_j(1) < -1$. If $b = -a^2$, then $M_j(t) = 1, T_j(t) = T_j(0)$, and $u(x,t) = u_0(x)$, for all $t$. Therefore, the condition $|b| \leq a^2$ is necessary and sufficient for the series

$$u(x,t) = \sum_{j=1}^{\infty} T_j(t) X_j(x) \tag{4.13}$$

to be a solution of the scalar BVP (4.1)-(4.3), with $A = a^2$ and $B = b$, if $u_0(x)$ is three times continuously differentiable. The coefficients $T_j(0)$ are given by

$$T_j(0) = \int_0^1 u_0(x) X_j(x) dx,$$

where $X_j(x) = \sqrt{2}\ sin(\pi j x)$ and $u_0(x) \in C^3[0,1]$ satisfies

$$u_0(0) = u_0(1) = 0.$$

The solution $T = 0$ of (4.6) is globally asymptotically stable as $t \to +\infty$ if and only if the eigenvalues $\lambda_r$ of the matrix $M(1)$ satisfy the inequalities

$$|\lambda_r| < 1, \qquad r = 1, \cdots, m. \tag{4.14}$$

If all eigenvalues of $A$ have positive real parts and $U_0(x) \in C^3[0,1]$, $\| A^{-1}B \| < 1$, then BVP (4.1)-(4.3) has a solution (4.13). This series and all its term-by-term derivatives converge uniformly.

Separation of variables in the equation with constant coefficients

$$u_{tt}(x,t) = a^2 u_{xx}(x,t) - b u_{xx}(x,[t]) \tag{4.15}$$

and boundary conditions (4.2) yields $X_j(x) = \sqrt{2}\ sin(\pi j x)$ and leads to the EPCA

$$T_j''(t) + a^2\pi^2 j^2 T_j(t) = b\pi^2 j^2 T_j([t]). \tag{4.16}$$

For brevity, omit the subindex $j$ and use the substitution $T'(t) = V(t)$, which changes (4.16) to a vector EPCA

$$w'(t) = Aw(t) + Bw([t]), \tag{4.17}$$

where $w = col(T, V)$ and

$$A = \begin{pmatrix} 0 & 1 \\ -a^2\pi^2 j^2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ b\pi^2 j^2 & 0 \end{pmatrix}.$$

Eq. (4.17) on the interval $n \le t < n+1$ becomes

$$w'(t) = Aw(t) + Bc_n, \qquad c_n = w(n)$$

with the solution

$$w(t) = M(t-n)c_n,$$

where

$$M(t) = e^{At} + (e^{At} - I)A^{-1}B. \tag{4.18}$$

Therefore, (4.17) with the initial condition $w(0) = c_0$ has on $[0, \infty)$ a unique solution given by the right-hand side of (4.11) where $M(t)$ is defined in (4.18).

For $b < 0$, the solution $w = 0$ of Eq. (4.17) is unstable. Indeed, computations show that

$$e^{At} = cos(\omega t)I + \omega^{-1} sin(\omega t)A$$

and

$$e^{At} - I = \begin{pmatrix} cos\ \omega t - 1 & \omega^{-1} sin\ \omega t \\ -\omega\ sin\ \omega t & cos\ \omega t - 1 \end{pmatrix}.$$

where $\omega = a\pi j$. Also

$$(e^{At} - I)A^{-1}B + \begin{pmatrix} b(1 - cos\ \omega t)/a^2 & 0 \\ (b\omega\ sin\ \omega t)/a^2 & 0 \end{pmatrix}$$

Hence,

$$M(t) = \begin{pmatrix} cos\ \omega t + ba^{-2}(1 - cos\ \omega t) & \omega^{-1} sin\ \omega t \\ (ba^{-2} - 1)\omega\ sin\ \omega t & cos\ \omega t \end{pmatrix}$$

and

$$det\ M(1) = 1 - \frac{b}{a^2} + \frac{b}{a^2} cos\ \omega.$$

The condition $b < 0$ implies $det M(1) > 1$ and shows that at least one of the eigenvalues $\lambda$ of $M(1)$ satisfies $|\lambda| > 1$. Therefore, $\| w(t) \| \to \infty$ as $t \to +\infty$, for some initial vector $c_0 \neq 0$.

For $b > a^2$, the solution $w = 0$ of Eq. (4.17) is unstable. Calculations give

$$det(M(1) - \lambda I) = \lambda^2 - 2\left(cos\ \omega + \frac{b}{a^2} sin^2 \frac{\omega}{2}\right)\lambda + 1 - \frac{b}{a^2} + \frac{b}{a^2} cos\ \omega$$

and the expressions $\lambda_1 = s + d, \lambda_2 = s - d$ for the eigenvalues $\lambda_1, \lambda_2$ of $M(1)$, where

$$s = \cos\omega + \frac{b}{a^2}\sin^2\frac{\omega}{2},\quad d^2 = \left(\frac{b}{a^2} - 1\right)\sin^2\omega + \frac{b^2}{a^4}\sin^4\frac{\omega}{2}.$$

The condition $b > a^2$ shows that $d^2 > 0$ and $\lambda_1 > 1$. The latter inequality implies $\| w(t) \| \to\infty$ as $t \to +\infty$, for some initial vector $c_0 \neq 0$.

The solution $w = 0$ of (4.17) is asymptotically stable as $t \to +\infty$ if and only if

$$0 < b < a^2, \tag{4.19}$$

and $\omega \neq 2\pi n, n = 0,1,2,\cdots$. The condition $d^2 < 0$, which means that the eigenvalues of $M(1)$ are complex, leads to

$$\cos^2\frac{\omega}{2} > \frac{b^2}{(2a^2 - b)^2},$$

whence

$$b < a^2\left(1 - \tan^2\frac{\omega}{4}\right) \text{ or } b < a^2\left(1 - \cot^2\frac{\omega}{4}\right).$$

Since $|\lambda_1| = |\lambda_2|$ and $\det M(1) = \lambda_1\lambda_2$, the inequality $|\lambda_1| < 1$ is equivalent to $\det M(1) < 1$, that is, to $b > 0$. Therefore, in the case of complex eigenvalues, a criterion for asymptotic stability is

$$0 < b < \max\left(a^2\left(1 - \tan^2\frac{\omega}{4}\right),\ a^2\left(1 - \cot^2\frac{\omega}{4}\right)\right).$$

The inequality $d^2 > 0$ in the case of distinct real eigenvalues leads to

$$b > \max\left(a^2\left(1 - \tan^2\frac{\omega}{4}\right),\ a^2\left(1 - \cot^2\frac{\omega}{4}\right)\right),$$

and the inequalities $\lambda_1 < 1,\ \lambda_2 > -1$ yield $b < a^2$. Hence, in this case a criterion of asymptotic stability is

$$\max\left(a^2\left(1 - \tan^2\frac{\omega}{4}\right),\ a^2\left(1 - \cot^2\frac{\omega}{4}\right)\right) < b < a^2.$$

Finally, if

$$b = \max\left(a^2\left(1 - \tan^2\frac{\omega}{4}\right),\ a^2\left(1 - \cot^2\frac{\omega}{4}\right)\right),$$

then $d = 0$ and $\lambda_1 = \lambda_2 = \cos\omega + ba^{-2}\sin^2\omega/2$, whence

$$\cos\omega < \lambda_1 < \cos^2\omega/2$$

and $|\lambda_1| < 1$. According to (4.14), this implies asymptotic stability and completes the proof of criterion (4.19).

If $b = a^2$, then $\lambda_1 = 1, \lambda_2 = \cos\omega$, and the solutions of (4.17) are bounded but not asymptotically stable. If $\omega = 2\pi n$, then $\lambda_1 = \lambda_2 = 1$, which leads to the existence of unbounded solutions for (4.17). If the coefficient $a$ is irrational, then (4.19) is a criterion of asymptotic stability of the solutions to (4.16) for all $j$, since recalling that $\omega = \omega_j = a\pi j$, we note that the equality $a\pi j = 2\pi n$ is impossible for any irrational $a$. For any rational $a$, there exist infinitely many integers $j$ such that the corresponding solutions $w_j(t)$ of (4.17) are unbounded.

Furthermore, each component of every solution of (4.17) oscillates if and only if either of the following conditions holds true:

(i) $b < max\left(a^2\left(1 - tan^2\frac{\omega}{4}\right), a^2\left(1 - cot^2\frac{\omega}{4}\right)\right)$,

(ii) $max\left(a^2\left(1 - tan^2\frac{\omega}{4}\right), a^2\left(1 - cot^2\frac{\omega}{4}\right)\right) < b < \frac{a^2}{2\, sin^2\frac{\omega}{2}}$ and $cos\,\omega < -\frac{1}{2}$.

In conclusion, it is worth noting that the asymptotic properties of (4.16) depend on the algebraic nature of the coefficient $a$. For $b < 0$, all solutions of (4.16) are unstable and oscillatory; for $b > a^2$ all solutions of (4.16) are unstable and nonoscillatory. These two cases hold true for both rational and irrational values of $a$. For

$$0 < b < max\left(a^2\left(1 - tan^2\frac{\omega}{4}\right), a^2\left(1 - cot^2\frac{\omega}{4}\right)\right),$$

all solutions of (4.16) are asymptotically stable and oscillatory, provided that $\omega \neq 2\pi n$. However, for any rational $a$, there exist infinitely many integers $j$ such that $\omega_j = 2\pi n$, which leads to the existence of unbounded solutions for (4.16). Furthermore, since $\omega = \omega_j = a\pi j$ the inequality $cos\,\omega < -1/2$ breaks down for infinitely many integers $j$. Therefore, under the above hypothesis (ii), there are infinitely many solutions of (4.16) which are asymptotically stable and oscillatory, as well as infinitely many solutions which are asymptotically stable and nonoscillatory ($\omega \neq 2\pi n$). Also, for $\omega \neq 2\pi n$ and $a^2/2sin^2(\omega/2) < b < a^2$, the solutions of (4.16) are asymptotically stable and nonoscillatory. Problems of this nature deserve further investigation. The following topics which are, in our opinion, of considerable interest either have not been explored at all or deserve deeper study.

(1) Partial differential equations with both constant and piecewise constant delays.

(2) Cauchy-Kovalevsky type existence-uniqueness theorems for partial differential equations with the argument $0 < \lambda(t) \leq t$ by using piecewise constant delays.

(3) Boundary and initial-value problems for PDE with alternately retarded and advanced piecewise continuous arguments.

(4) Parabolic PDE of neutral type with piecewise constant time.

(5) Bounded solutions of nonlinear parabolic equations with piecewise continuous arguments.

(6) Boundary and initial-value problems for the wave equation with the argument $[\lambda t/h]h$, $0 < \lambda < 1$, $h > 0$.

(7) Bounded solutions of nonlinear hyperbolic equations with the argument $[\lambda t/h]h$.

(8) Loaded partial differential-difference equations.

In conclusion, we note that parabolic equations with unbounded piecewise constant delay were studied in [13], and the first monograph on equations with piecewise continuous arguments was published in 1993 [14].

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