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ON GENERIC SUBMANIFOLDS OF A LOCALLY CONFORMAL KAHLER MANIFOLD WITH PARALLEL CANONICAL STRUCTURES

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ABSTRACT. The study of CR-submanifolds of a Kähler manifold was initiated by Bejancu [1]. Since then many papers have appeared on CR-submanifolds of a Kähler manifold. Also, it has been studied that generic submanifolds of Kähler manifolds [2] are generalisations of holomorphic submanifolds, totally real submanifolds and CR-submanifolds of Kähler manifolds. On the other hand, many examples of generic surfaces in C^2 which are not CR-submanifolds have been given by Chen [3] and this leads to the present paper where we obtain some necessary conditions for a generic submanifolds in a locally conformal Kahler manifold with four canonical strucrures, denoted by P,F,t and f, to have parallel P,F and t. We also prove that for a generic submanifold of a locally conformal Kahler manifold, F is parallel iff t is parallel.

KEY WORDS AND PHRASES. CR-submanifold, generic submanifold and locally conformal Kähler manifold.
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1. INTRODUCTION.

Let $\bar{M}$ be an almost Hermitian manifold and Ω be the fundamental

2-form given by $g(JX,Y) = \Omega(X,Y)$, g being the Hermitian metric. Then $\bar{M}$ is called a locally conformal Kähler manifold (l.c.k.) if there is a closed 1-form ω, called the Lee-form on $\bar{M}$, such that $d\Omega = \Omega \wedge \omega$, where d and $\wedge$ denote exterior derivative operator and wedge product, respectively [4]. Let $\bar{M}$ be a l.c.k. manifold. Then the vector field B (Lee field of $\bar{M}$) is defined as $g(X,B) = \omega(X)$. Now suppose that $\bar{\nabla}$ be the Levi-Civita connection of g. Then, we have [4]

$$\tilde{\nabla}_X Y = \bar{\nabla}_X Y - \frac{1}{2}\Big[\omega(X)Y + \omega(Y)X - g(X,Y)B\Big], \tag{1.1}$$

where $\tilde{\nabla}$ is a torsionless linear connection on $\bar{M}$ called the Weyl connection of g.

The following result is known [4].

THEOREM. The almost Hermitian manifold $\bar{M}$ is a l.c.k. manifold if and only if there is a closed 1-form ω on $\bar{M}$ such that the Weyl connection be almost complex i.e. $\tilde{\nabla}J=0$.

DEFINITION. Let M be a submanifold of a l.c.k. manifold $\bar{M}$. The holomorphic tangent spaces to M at $x\in M$ are defined as $D_x = T_xM \cap JT_xM$, where D_x is the maximal complex subspace of T_xM. If the dimension of D_x is constant along M and it defines a differentiable distribution on M, then M is called a generic submanifold of $\bar{M}$. We call D_x the holomorphic distribution and the orthogonal complementary distribution $D_x^{\perp}$ is called purely real distribution [2]. A generic submanifold M of a l.c.k. manifold is a CR-submanifold [1] if the orthogonal complementary distribution $D_x^{\perp}$ of D_x in T_xM is totally real i.e. $JD_x^{\perp} \subset T_x^{\perp}M$, where $T_x^{\perp}M$ is the normal space of M at x.

Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$ and let ∇, $\tilde{\nabla}$ be covariant differentiation on M induced by $\bar{\nabla}$ and $\tilde{\nabla}$, respectively. Then Gauss and Weingarten formulae w.r.t. $\bar{\nabla}$ and $\tilde{\nabla}$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{1.2}$$

$$\tilde{\nabla}_X Y = \tilde{\nabla}_X Y + \tilde{h}(X,Y), \quad \tilde{\nabla}_X N = -\tilde{A}_N X + \tilde{\nabla}_X^{\perp} N \tag{1.3}$$

for any vector fields X,Y tangent to M and N normal to M. Here $h(\tilde{h})$ is the second fundamental form of M with respect to $\bar{\nabla}(\tilde{\nabla})$ and $\nabla^{\perp}(\tilde{\nabla}^{\perp})$ is the normal connection. Moreover,

$$g(A_N X,Y) = g(h(X,Y),N). \tag{1.4}$$

The transforms JX and JN of X and N by J are decomposed into tangential and normal parts as

$$JX = PX + FX, \tag{1.5}$$

$$JN = tN + fN, \tag{1.6}$$

where P and f are endomorphisms of TM and $T^{\perp}M$ and F and t are normal bundle valued 1-form on TM and tangent bundle valued 1-form on $T^{\perp}M$, respectively. For Lee vector field B tangent to M, we put

$$B_x = (B_1)_x + (B_2)_x, \qquad x \in M \tag{1.7}$$

where $(B_1)_x$ and $(B_2)_x$ are the tangential and normal components of B, respectively.

The following relations hold for a generic submanifold [2].

$$t(T^{\perp}M) = D^{\perp},\ PD = D \text{ and } PD^{\perp} \subseteq D. \tag{1.8}$$

2. **SOME RESULTS.**

We define the covariant differentiation of P,F,t and f as follows:

$$(\bar{\nabla}'_X P)Y = \nabla_X(PY) - P\nabla_X Y, \tag{2.1}$$

$$(\bar{\nabla}'_X F)Y = \nabla^{\perp}_X(FY) - F\nabla_X Y, \tag{2.2}$$

$$(\bar{\nabla}'_X t)N = \nabla_X tN - t\nabla_X N, \tag{2.3}$$

$$(\bar{\nabla}'_X f)N = \nabla^{\perp}_X fN - f\nabla^{\perp}_X N \tag{2.4}$$

for any vector fields X and Y tangent to M and any vector field N normal to M.

We say that P (respectively f, F or t) is parallel if $\bar{\nabla}' P = 0$ (respectively $\bar{\nabla}' f = 0$, $\bar{\nabla}' F = 0$ or $\bar{\nabla}' t = 0$).

Using (1.1), (1.2) and (1.3), we have

LEMMA 2.1. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$. Then

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}\Big[\omega(X)Y + \omega(Y)X - g(X,Y)B_1\Big], \tag{2.5}$$

$$\tilde{h}(X,Y) = h(X,Y) + \frac{1}{2}\, g(X,Y)B_2, \tag{2.6}$$

$$\tilde{A}_N X = A_N X + \frac{1}{2}\,\omega(N)X, \tag{2.7}$$

$$\tilde{\nabla}^{\perp}_X N = \nabla^{\perp}_X N - \frac{1}{2}\,\omega(X)N \tag{2.8}$$

for any vector fields X, Y tangent to M and any vector field N normal to M.

The following result is known [4].

LEMMA A. The holomorphic distribution D is integrable iff

$$g(\tilde{h}(X,JY),FZ) = g(\tilde{h}(JX,Y),FZ)$$

for any vector fields $X, Y \in D$ and $Z \in D^{\perp}$.

Using (2.6) in Lemma A, we have

COROLLARY 2.2. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$. Then the holomorphic distribution D is integrable iff

$$g(h(X,JY) - h(JX,Y) + \Omega(X,Y)B_2, FZ) = 0$$

for any $X, Y \in D$ and $Z \in D^{\perp}$.

From (1.3),(1.5) and (1.6), on comparing the tangential and normal components, we obtain

LEMMA 2.3. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$. Then

$$\underset{\sim}{\nabla}_X PY - \tilde{A}_{FY} X = P\underset{\sim}{\nabla}_X Y + t\tilde{h}(X,Y), \tag{2.9}$$

$$\tilde{h}(X,PY) + \tilde{\nabla}^{\perp}_X FY = F\underset{\sim}{\nabla}_X Y + f\tilde{h}(X,Y) \tag{2.10}$$

for any vector fields X,Y tangent to M.

3. GENERIC SUBMANIFOLDS WITH $\bar{\nabla}' P=0$.

We now assume that the canonical structure P is parallel on a generic submanifold of a l.c.k. manifold $\bar{M}$ i.e., $\bar{\nabla}' P=0$. Then from (2.5), we have

$$\underset{\sim}{\nabla}_X PY - P\underset{\sim}{\nabla}_X Y = \frac{1}{2}\left[\omega(Y)PX - \omega(PY)X + g(X,PY)B_1 - g(X,Y)PB_1\right]$$

for all vector fields X,Y tangent to M.

Using (2.9) in the above equation, we get

$$t\tilde{h}(X,Y) + \tilde{A}_{FY} X = \frac{1}{2}\left[\omega(Y)PX - \omega(PY)X + g(X,PY)B_1 - g(X,Y)PB_1\right] \tag{3.1}$$

which, in view of (1.4) and (1.8), give

$$2g(\tilde{h}(X,Z),FY) = \omega(Y)g(PX,Z) - \omega(PY)g(X,Z)$$

$$+g(X,PY)g(B,Z) - g(X,Y)g(PB,Z) \tag{3.2}$$

for all $Z\in D$ and $X,Y,B\in TM$.

Thus, for $X,Z\in D$ and $Y\in D^{\perp}$, equation (3.2) gives us the following:

PROPOSITION 3.1. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$ such that $B\in D$. If P is parallel, then the holomorphic distribution D is integrable, that is,

$$g(\tilde{h}(D,D),FD^{\perp}) = 0.$$

Similarly, for $X,Y\in D^{\perp}$ and $Z\in D$, we have, from (3.2), the following:

PROPOSITION 3.2. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$ such that $B\in D^{\perp}$. If P is parallel, then

$$g(\tilde{h}(D,D^{\perp}),FD^{\perp}) = 0.$$

We now assume that the Lee field B is normal to M. Then from (2.6), (2.7), (3.1) and using the fact that $t(T^{\perp}M)=D^{\perp}$, we have

$$th(X,Y)+\frac{1}{2}g(X,Y)B_2 = -A_{FY}X - \frac{1}{2}\omega(FY)X \tag{3.3}$$

for $X\in D$ and $Y\in D^{\perp}$. Then, for $Z\in D$ and using (1.6) in (3.3), we get

$$h(X,Z) = -\frac{1}{2}g(X,Z)B_2$$

whence

$$h(X,JZ) - h(JX,Z) = -g(X,JZ)B_2$$

which can be written as

$$g(h(X,JZ)-h(JX,Z)+\Omega(X,Z)B_2,FY) = 0.$$

Consequently, using Corollary (2.1), we have

PRPOPSITION 3.3. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$ with B normal to M. If P is parallel, then the holomorphic distribution D is integrable.

4. GENERIC SUBMANIFOLDS WITH $\bar{\nabla}'F=0$

In this section, we obtain some results assuming that F is parallel.

PROPOSITION 4.1. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$. If F is parallel, then distribution D is integrable and leaf of D is totally geodesic in M.

PROOF. Since F is parallel, we have from (2.2) that for all $X,Y \in D$, $F\nabla_X Y = 0$. Consequently, $\nabla_X Y\in D$ for any $X,Y\in D$, which shows that the distribution D is integrable and leaf of D is totally geodesic in M.

PROPOSITION 4.2. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$. If F is parallel, then

$$A_{fN}X+A_NPX = \frac{1}{2}\left[g(B_2,N)X-g(B_2,N)PX-\omega(X)tN\right]$$

for any vector field X tangent to M and N normal to M.

PROOF. Using $(\tilde\nabla_X J)Y=0$, the equations (1.3) and (1.5), we get

$$\underset{\sim}{\nabla}_X PY + \tilde h(X,PY)-\tilde A_{FY}X+\tilde\nabla^{\perp}_X FY = J(\underset{\sim}{\nabla}_X Y+\tilde h(X,Y)),$$

for all X tangent to M.

Using Lemma A and the relation $\tilde\nabla^{\perp}_X N = \nabla^{\perp}_X N- \frac{1}{2}\omega(X)N$, we get that

$$\nabla_X PY - \frac{1}{2}\,\omega(PY)X + \frac{1}{2}\,g(X,PY)(B_1+B_2) + h(X,PY) -A_{FY}X - \frac{1}{2}\,\omega(FY)X + \nabla^{\perp}_X FY$$

$$= P\nabla_X Y + F\nabla_X Y+th(X,Y)+fh(X,Y) - \frac{1}{2}\,\omega(Y)(PX+FX) + \frac{1}{2}\,g(X,Y)(B_1+B_2)$$

for any vector fields X,Y tangent to M.

Comparing the normal components on both sides of the above equation, we get

$$(\bar\nabla'_X F)(Y) = fh(X,Y)-h(X,PY)+ \frac{1}{2}\left[g(X,Y)B_2-g(X,PY)B_2-\omega(Y)FX\right]$$

Hence, for any vector field N normal to M, we get

$$g((\bar\nabla'_X F)Y,N)= -g\left(A_{fN}Y+A_N PY,X\right)$$

$$+\frac{1}{2}\left[g(X,Y)g(B_2,N)-\omega(Y)g(FX,N)-g(X,PY)g(B_2,N)\right].$$

Thus, if F is parallel, we get the result.

From the above result, we immediately get the following :

COROLLARY 4.3. Let M be a generic submanifold of a l.c.k. manifold $\bar M$ and let B∈D. If P is parallel, then

(a) $A_{fN}X - A_N PX = -\frac{1}{2}\,\omega(X)tN$, for any vector field X tangent to M, and

(b) $g(Jh(X,Y),N) = 0$, or $g(Jh(D,D^{\perp}),N) = 0$ for all $X \in D^{\perp}$ and $Y \in D$.

5. GENERIC SUBMANIFOLDS WITH $\bar{\nabla}'t = 0$

Now, we consider generic submanifold with parallel t. Then, using $(\tilde{\nabla}_X J)N = 0$, the equations (1.3) and (1.6), we get

$$\tilde{\nabla}_X tN + \tilde{h}(X,tN) - \tilde{A}_{fN}X + \tilde{\nabla}_X fN = J(-\tilde{A}_N X + \tilde{\nabla}^{\perp}_X N),$$

for all X tangent to M and N normal to M.

Using Lemma A in the above equation, we get

$$\nabla_X tN - \frac{1}{2}\,\omega(tN)X + \frac{1}{2}\,g(X,tN)B_1 + h(X,tN) + \frac{1}{2}\,g(X,tN)B_2$$

$$- A_{fN}X - \frac{1}{2}\,\omega(fN)X + \nabla^{\perp}_X fN = t\nabla^{\perp}_X N + f\nabla^{\perp}_X N - PA_N X$$

$$- FA_N X - \frac{1}{2}\,\omega(N)(PX+FX)$$

for any vector field X tangent to M and N normal to M.

Comparing tangential components, we get

$$(\bar{\nabla}'_X t)N = A_{fN}X - PA_N X + \frac{1}{2}\left[\omega(JN)X - \omega(N)PX - g(X,tN)B_1\right] \tag{5.1}$$

for any vector field X tangent to M and N normal to M.

Thus we have

PROPOSITION 5.1. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$. If t is parallel, then

$$A_{fN}X - PA_N X = \frac{1}{2}\left[\omega(N)PX - \omega(JN)X + g(X,tN)B_1\right]$$

for any vector field X tangent to M and N normal to M.

The following is immediate.

COROLLARY 5.2. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$ such that $B \in D$. If t is parallel, then

(a) $A_{fN}X - PA_N X = \frac{1}{2} g(X,tN)B_1$, for any X tangent to M, and

(b) $g(Jh(X,Y),N) = 0$, or $g(Jh(D,D^{\perp}),N) = 0$ for any $X \in D$ and $Y \in D^{\perp}$

PROPOSITION 5.3. Let M be a generic submanifold of a l.c.k. manifold $\bar{M}$. Then F is parallel if and only if t is parallel.

PROOF. Let t be parallel. Then, from (5.1) we have

$$g(A_{fN}X,Y)-g(PA_N X,Y) + \frac{1}{2} g(B,JN)g(X,Y)$$

$$- \frac{1}{2} g(X,tN)g(B_1,Y)- \frac{1}{2} g(PX,Y)g(B_2,N) = 0$$

for any vector fields X,Y tangent to M and any vector field N normal to M. The above equation becomes

$$-g(fh(X,Y),N)+g(h(X,PY),N) - \frac{1}{2} g(B_2,N)g(X,Y)$$

$$+ \frac{1}{2} g(FX,N)g(B_1,Y) - \frac{1}{2} g(B_2,N)g(PX,Y) = 0$$

which is equivalent to

$$fh(X,Y)-h(X,PY) + \frac{1}{2}\left[g(X,Y)B_2-g(X,PY)B_2-\omega(Y)FX\right] = 0$$

that is,

$$\bar{\nabla}'F = 0.$$

REMARK

1. Let $M^{\perp}$ be any complex submanifold in complex number space C^r and M be any purely real submanifold of dimension p in the complex number space C^p. Then it can be verified that the Riemannian product $M^{\perp} \times M^{\perp}$is a generic submanifold in C^{r+p} satisfying $\bar{\nabla}'P = 0$ and $\bar{\nabla}'F = 0$.

2. It is to be noted that many examples of generic surfaces in C^2 which are not CR-submanifolds can be found in Chen's book [3].

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