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A COMMON FIXED POINT THEOREM FOR A SEQUENCE OF FUZZY MAPPINGS

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ABSTRACT. We obtain a common fixed point theorem for a sequence of fuzzy mappings, satisfying a contractive definition more general than that of Lee, Lee, Cho and Kim [2].

Let (X,d) be a complete linear metric space. A fuzzy set A in X is a function from X into [0,1]. If $x \in X$, the function value $A(x)$ is called the grade of membership of X in A. The α-level set of $A, A_\alpha := \{x : A(x) \geq \alpha, \text{ if } \alpha \in (0,1]\}$, and $A_0 := \overline{\{x : A(x) > 0\}}$. $W(X)$ denotes the collection of all the fuzzy sets A in X such that A_α is compact and convex for each $\alpha \in [0,1]$ and $\sup_{x\in X} A(x) = 1$. For $A, B \in W(X), A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$. For $A, B \in W(X), \alpha \in [0,1]$, define

$$P_\alpha(A,B) = \inf_{x\in A_\alpha, y\in B_\alpha} d(x,y), P(A,B) = \sup_\alpha P_\alpha(A,B), D(A,B) = \sup_\alpha d_H(A_\alpha,B_\alpha),$$

where d_H is the Hausdorff metric induced by the metric d. We note that P_α is a nondecreasing function of α and D is a metric on $W(X)$.

Let X be an arbitrary set, Y any linear metric space. F is called a fuzzy mapping if F is a mapping from the set X into $W(Y)$.

In earlier papers the author and Bruce Watson, [3] and [4], proved some fixed point theorems for some mappings satisfying a very general contractive condition. In this paper we prove a fixed point theorem for a sequence of fuzzy mappings satisfying a special case of this general contractive condition. We shall first prove the theorem, and then demonstrate that our definition is more general than that appearing in [2].

Let D denote the closure of the range of d. We shall be concerned with a function Q, defined on d and satisfying the following conditions:

(a) $0 < Q(s) < s$ for each $s \in D\backslash\{0\}$ and $Q(0) = 0$,

(b) Q is nondecreasing on D, and

(c) $g(s) := s/(s - Q(s))$ is nonincreasing on $D\backslash\{0\}$.

LEMMA 1. [1] Let (X,d) be a complete linear metric space, F a fuzzy mapping from X into $W(X)$ and $x_0 \in X$. Then there exists an $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.

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THEOREM 1. Let g be a nonexpansive selfmap of X, (X,d) a complete linear metric space. Let $\{F_i\}$ be a sequence of fuzzy mappings from X into $W(X)$ satisfying: For each pair of fuzzy mappings F_i, F_j and for any $x \in X, \{u_x\} \subset F_i(x)$, there exists a $\{v_y\} \subset F_j(y)$ for all $y \in X$ such that

$$D(\{u_x, v_y\}) \le Q(m(x,y)), \tag{1}$$

where

$$m(x,y) := \max\{d(g(x),g(u_x)), d(g(y),g(v_y)), \Big[d(g(y),g(u_x)) + d(g(x),g(v_y))\Big]/2, d(g(x),g(y))\} \tag{2}$$

and Q satisfies (a) - (c). Then there exists a $p \in \cap_{i=1}^{\infty} F_i(p)$.

PROOF. Let $x_0 \in X$. Then we can choose $x_1 \in X$ such that $\{x_1\} \subset F_1(x_0)$ by Lemma 1. From the hypothesis, there exists an $x_2 \in X$ such that $\{x_2\} \subset F_2(x_1)$ and, from (1),

$$\begin{aligned} D(\{x_1\},\{x_2\}) &\le Q(m(x_1,x_2)) \\ &< \max\{d(g(x_0),g(x_1)), d(g(x_1,g(x_2)), \\ &\quad [d(g(x_1),g(x_1)) + d(g(x_0),g(x_2))]/2, d(g(x_0),g(x_1))\} \\ &\le \max\{d(x_0,x_1), d(x_1,x_2), [d(x_1,x_1) + d(x_0,x_2)]/2, d(x_0,x_1)\}, \end{aligned}$$

since g is nonexpansive.
Inductively, we obtain a sequence $\{x_n\}$ such that $x_{n+1} \subset F_{n+1}(x_n)$ and

$$\begin{aligned} D(\{x_n\},\{x_{n+1}\}) &\le Q(m(x_n,x_{n+1})) \\ &< \max\{d(x_{n-1},x_n), d(x_n,x_{n+1}), [d(x_n x_n) + dx_{n-1},x_{n+1})]/2, \\ &\quad d(x_{n-1},x_n)\}. \end{aligned} \tag{3}$$

Since $D(\{x_n\},\{x_{n+1}\}) = d(x_n,x_{n+1})$, it follows from (3) that $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$. Using this fact back in (2), we obtain that $d(x_n,x_{n+1}) \le d(x_{n-1},x_n)$. Substituting into (3) we obtain

$$d(x_n,x_{n+1}) < Q(d(x_{n-1},x_n)) < Q^2(d(x_{n-2},x_{n-1}) < \cdots < Q^n(d(x_0,x_1)).$$

From Lemma 2 of [3], $\lim Q^n(d(x_0,x_1)) = 0$. To show that $\{x_n\}$ is Cauchy, choose N so large that $Q^n(d(x_0,x_1)) \le 1/2$ for all $n > N$. Then, for $m > n > N$,

$$d(x_n,x_m) \le \sum_{i=n}^{m-1} d(x_i,x_{i+1}) \le \sum_{i=n}^{m-1} Q^i(d(x_0,x_1)) \le \sum_{i=n}^{m-1} \left(\frac{1}{2}\right)^i < \frac{1}{2^n}\left(1 - \frac{1}{2}\right),$$

and $\{x_n\}$ is Cauchy, hence convergent. Call the limit p.

Let F_m be an arbitrary member of the sequence $\{F_i\}$. Since $\{x_n\} \subset F_n(x_{n-1})$, there exists a $v_n \in X$ such that $\{v_n\} \subset F_m(p)$ for all n and

$$\begin{aligned} D(\{x_n\},\{v_n\}) &\le Q(m(x_n,v_n)) \\ &= Q\big(\max\{d(g(x_{n-1}),g(x_n)), d(g(p),g(v_n)), \\ &\quad [d(g(x_{n-1}),g(v_n)) + d(g(p),g(x_n))]/2, d(g(x_{n-1}),g(p))\}\big) \\ &\le Q\big(\max\{d(x_{n-1},x_n), d(p,v_n), [d(x_{n-1},v_n) + d(p,x_n)]/2, d(x_{n-1},p)\}\big). \end{aligned}$$

Suppose that $\lim v_n \ne p$. Taking the limit as $n \to \infty$ yields, since Q is continuous (Lemma 1 of [3]),

$$\limsup d(p,v_n) \le Q(\limsup d(p,v_n) < \limsup d(p,v_n),$$

a contradiction. Therefore $\lim v_n = p$. Since $F_m(p) \in W(X), F_m(p)$ is upper semicontinuous and therefore $\limsup [F_m(p)](v_n) \leq [F_m(p)](p)$. Since $\{v_n\} \subset F_m(p)$ for all $n, [F_m(p)](p) = 1$. Hence $\{p\} \subset F_m(p)$. Since F_m is arbitrary, $\{p\} \subset \cap_{i=1}^{\infty} F_i(p)$.

The contractive definition of [2] is the following :

$$D(\{u_x\},\{v_y\}) \leq a_1 d(g(x),g(u_x)) + a_2 d(g(y),g(v_y)) + a_3 d(g(y),g(u_x)) + a_4 d(g(x),g(v_y)) + d(g(x),g(y)) \tag{4}$$

where each a_i is nonnegative, $\sum_{i=1}^{5} a_i < 1$, and $a_3 \geq a_4$.

In (4), if one interchanges the roles of x and y one obtains

$$D(\{v_y\},\{u_x\}) \leq a_1 d(g(y),v_y)) + a_2 d(g(x),g(u_x)) + a_3 d(g(x),g(v_y)) + a_4 d(g(y),g(u_x)) + a_5 d(g(y),g(x)). \tag{5}$$

Adding (4) and (5) yields

$$D(\{u_x\},\{v_y\}) \leq \alpha_1 d(g(x),g(u_x)) + \alpha_2 d(g(y),g(v_y)) + \alpha_3 d(g(y),g(u_x)) + \alpha_4 d(g(x),g(v_y)) + \alpha_5 d(g(x),g(y)), \tag{6}$$

where $\alpha_1 = \alpha_2 = (a_1 + a_2)/2, \alpha_3 = \alpha_4 = (a_3 + a_4)/2$, and $\alpha_5 = a_5$. In turn, (6) implies that

$$\begin{aligned} D(\{u_x\},\{v_y\}) &\leq (\alpha_1 + \alpha_2)\max\{d(g(x),g(u_x)), d(g(y),g(v_y))\} + \\ &\quad \alpha_3 [d(g(y),g(u_x)) + d(g(x),g(v_y))]/2 + \alpha_5 d(g(x),g(y)) \\ &\leq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5) m(x,y) = \left(\sum_{i=1}^{5} a_i\right) m(x,y) = hm(x,y), \end{aligned} \tag{7}$$

say.

(7) is the special case of (1) with $Q(s) = hs$. Consequently Theorem 3.1 of [2], as well as the corollaries, are special cases of the theorem of this paper.

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