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# QUASIMINIMAL DISTAL FUNCTION SPACE AND ITS SEMIGROUP COMPACTIFICATION

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ABSTRACT. Quasiminimal distal function on a semitopological semigroup is introduced. The concept of right topological semigroup compactification is employed to study the algebra of quasiminimal distal functions. The universal mapping property of the quasiminimal distal compactification is obtained.

KEY WORDS AND PHRASES. Quasiminimal distal flows and distal functions, right topological semigroup compactification, universal mapping property.
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In a recent paper, we generalized the notion of distal flows and distal functions on an arbitrary semitopological semigroup S and studied their function spaces [3]. The study was an extension of Junghenn's work on distal functions and their semigroup compactifications [2]. The impetus for the author's work in this area is the study by Berglund et al. in which the authors have introduced several kinds of function spaces which are admissible $C^*$-subalgebras of C(S) and studied these function spaces through semigroup compactifications [1]. Interestingly, these compactifications possess certain universal mapping properties, universal in the sense that the compactifications are maximal with respect to these properties. This approach of studying function spaces through semigroup compactifications enables one to take advantage of the universal mapping properties the compactifications enjoy. For example, in the case of distal functions on a semigroup S, Junghenn has effectively used the universal mapping properties to show that the space of weakly almost periodic distal functions on S is the direct sum of the algebra of strongly almost periodic functions and two ideals of weakly almost "flight" functions on S [2]. In this paper, we introduce the quasiminimal distal function space QMD(S), and show that this space is an admissible $C^*$-subalgebra of C(S). As a consequence, we obtain the QMD(S)-compactification of S which is the spectrum of QMD(S). We then characterize this semigroup compactification in terms of the universal mapping property the compactification possesses.

## 1. PRELIMINARIES

Throughout our work, S denotes a semitopological semigroup with Hausdorff topology and C(S) denotes the $C^*$-algebra of all bounded complex-valued continuous functions on S. A subspace F of C(S) is translation invariant if $L_sF \subset F$ and $R_sF \subset F$

where the operators $L_s$ and $R_s$ on C(S) are defined by $L_sf(t) = f(st)$ and $R_sf(t) = f(ts)$ for s, t $\in$ S and f $\in$ C(S). We define $T_x.F \to B$ by $T_xf(\cdot) = x(L_{(\ )}f)$ where f $\in$ F and x $\in$ the spectrum of F which is the space of nonzero continuous complex homomorphisms on F. A $C^*$-subalgebra F of C(S) is called <u>admissible</u> if it is translation invariant, contains the constant functions, and $T_xF\subset F$ for every x $\in$ spectrum of F. A pair $(X,\alpha)$ is called a <u>right topological compactification</u> of S if X is a compact right topological semigroup and $\alpha:S \to X$ is a continuous homomorphism with dense image such that the mapping $x \to \alpha(s)x:X \to X$ is continuous for each s $\in$ S. It is called an <u>F-compactification</u> of S if F is an admissible subalgebra of C(S) and $\alpha^*$ C(X) = F. We call X the <u>phase space</u> of the compactification. The set of all multiplicative means on F, denoted by MM(F), is a $w^*$-compact right topological semigroup with binary operation defined in MM(F) by $xy = xoT_y$. If $e:S \to MM(F)$ is the evaluation map $(e(s)(f) = f(s))$, then e(S) is dense in MM(F) and the pair (MM(F), e) is an F-compactification of S. For a fixed F, all F-compactifications are isomorphic both algebraically and topologically. Thus, there is a unique F-compactification of S. We call (MM(F), e), the <u>canonical</u> F-compactification of S. The map $T_x:F \to B$ has the following properties (for details see [1. I·4]). For s $\in$ S, f $\in$ F and x, y $\in$ X, a) $T_{xy} = T_xoT_y$ b) $[e(s)x](f) = x\ (L_sf)$ c) $[x\ e(s)](f) = x\ (R_sf)$. A right topological compactification $(X,\alpha)$ of S is said to be <u>maximal</u> (<u>universal</u>) with respect to a property P if $(X,\alpha)$ has the property P and if $(Y,\beta)$ is a right topological compactification of S with property P, then $\alpha$ factors $\beta$ in the sense that $\alpha = \lambda o\beta$ for some continuous homomorphism $\lambda:X\to Y$. A very useful admissible subalgebra of C(S) is LMC(S) = $\{f \in C(S) \mid s \to x\ (L_sf)$ is continuous for every $x \in$ MM C(S)$\}$. It is the largest admissible subalgebra of C(S) and the LMC(S)-compactification of S is maximal with respect to the property that it is a right topological compactification of S [1].

If X is a compact topological space and $\pi:S \to X^X$ is a continuous homomorphism such that $\pi(s):X \to X$ is continuous for each s $\in$ S, then the triple (S, X, $\pi$) is called a <u>flow</u>. For convenience, we write sx for $\pi(s)(X)$. The flow is called <u>distal</u> (respectively, <u>quasidistal</u>) if, whenever x,y,$\in$X such that $\lim_i s_i x = \lim_i s_i y$ for some net $(s_i)$ in S, then x = y (respectively, sx = sy for every s $\in$ S). A function f $\in$ LMC(S) is called a distal (respectively, a quasidistal) function if the flow (S, $Z_f$, $\pi$), where $Z_f$ is the closure of $R_Sf$ in the topology of pointwise convergence on C(S) and $\pi(s):R_Sf|_{Z_f}$ is distal (respectively, a quasidistal). It is called a <u>minimal</u> function if g $\in$ $Z_f$ implies f $\in$ $Z_g$. Junghenn has shown that the set of all minimal distal functions, MD(S), is an admissible subalgebra of C(S) and that MD(S)-compactification $(Y,\beta)$ is maximal with respect to the property that Y is left simple [2]. The author has proved analogous results for quasidistal functions, QD(S). For more on these functions and related theorems, the reader is referred to [3,4].

## 2. QUASIMINIMAL DISTAL FUNCTIONS

DEFINITION 2·1. A function of f $\in$ LMC(S) is <u>quasiminimal</u> if for every g $\in$ $Z_f$, there exists x $\in$ X such that $T_uf = T_{ux}g$ for every u $\in$ X where X is the phase space of the canonical LMC(S)-compactification $(X,\alpha)$.
We denote the set of all quasiminimal functions by QM(S).

PROPOSITION 2·2 Let $f \in LMC(S)$ and $(X,\alpha)$ denote the canonical LMC(S)-compactification of S. Then $f \in QM(S)$ if and only if for each $y \in X$, there exists $x \in X$ (depending on y and f) such that $T_{uxy}f = T_uf$ for every $u \in X$.

PROOF. Necessity: Assume f is quasiminimal. Let $y \in X$. Then $T_yf \in \{T_xf \mid x \in X\} = Z_f$ By definition, there exists $x \in X$ such that $T_uf = T_{ux}T_yf = T_{uxy}f$ for every $u \in X$.

Sufficiency: Let $g \in Z_f = \{T_xf \mid x \in X\}$ which implies that $g = T_yf$ for some $y \in X$. For this y, there exists $x \in X$ such that $T_uf = T_{uxy}f = T_{ux}T_yf = T_{ux}g$ for every $u \in X$. Then by definition, $f \in QM(S)$.

DEFINITION: 2·3. A function of $\in$ LMC(S) is said to be quasiminimal distal if $f \in QM(S)\cap Q\mathcal{D}(S)$. We write $QM\mathcal{D}(S)$ for $QM(S)\cap QD(S)$.

PROPOSITION 2 4. Let $f \in LMC(S)$ and $(X,\alpha)$ denote the canonical LMC(S)-compactification of S. $f \in QMD(S)$ if and only if $f \in QD(S)$ and $v(f) = ve(f)$ for every $v \in \overline{X^2}$, $e \in E(x)$

Remark 2·5. Before we present the proof of this proposition, we state here without proof the author's characterization of quasidistal functions QD(S) A function $f \in LMC(S)$ is quasidistal, that is $f \in QD(S)$ if and only if $uev(f) = uv(f)$ for every $u \in \overline{X^2}$, $v \in X$ and $e \in E(X)$ where X is the phase space of the LMC(S)-compactification of S.

PROOF. Necessity. Assume $f \in QMD(S) = Q\mathcal{D}(S)\cap QM(S)$. Let $e \in E(X) \subseteq X$. By Proposition 1, there exists $x \in X$ such that $T_{uxe}f = T_uf$ for every $u \in X$ (2.2). Let $v \in X^2$; then $v = v_1v_2$ and $T_{v_2xe}f = T_{v_2}f$ by (2.2). Therefore, $v_1(T_{v_2xe}f) = v_1(T_{v_2}f)$ which implies that $v_1v_2xe(f) = v_1v_2(f)$. Now, $v(f) = v_1v_2(f) = v_1v_2xe(f) = v_1v_2exe(f)$ (by 2·1) $= v_1(T_{v_2exe}f) = v_1(T_{v_2e}f)$ (by 2·2) $= v_1v_2e(f) = ve(f)$. That the same result holds for $v \in \overline{X^2}$ follows from the right topological property of X.

Sufficiency. It suffices to prove that $f \in QM(S)$. We recall that X has the smallest ideal K(X) and for each $e \in E(K(X))$, eXe is a maximal subgroup of X with identity e [5]. Let $y \in X$. There exists $x \in X$ such that $(exe)(eye) = e$ which implies $exeye = e$. Let $u \in X$. For any $v \in X$, $v(T_{uxy}f) = vuxy(f) = vuexy(f) = vuexey(f)$ (the last two equalities by 2·5) $= vuexeye(f)$ (by hypothesis) $= vue(f) = vu(f)$ (by hypothesis) $= v(T_uf)$. Since v was arbitrary, $T_{uxy}f = T_uf$. Therefore by Proposition 2·2, $f \in QM(S)$

THEOREM 2·6. QMD(S) is one admissible subalgebra of C(S).

PROOF. Let $(X,\alpha)$ denote the canonical LMC(S)-compactification of S. Linearity of the space QMD(S) is immediate from Proposition 2·4. Let $f \in QMD(S)$, $v \in \overline{X^2}$ and $e \in E(X)$. For $s \in S$, $ve(L_sf) = \alpha(s)ve(f) = \alpha(s)v(f)$ (Prop. 2·4) $= v(L_sf)$ which implies that $L_sf \in QMD(S)$. Now $ve(R_sf) = ve\alpha(s)(f) = v\alpha(s)(f)$ (Remark 2·5) $= v(R_sf)$ which proves that $R_sf \in QMD(S)$. Thus QMD(S) is translation invariant. That QMD(S) is an algebra follows from the fact that X is the set of all multiplicative means on LMC(S). Trivially, $v(1) = ve(1)$. So QMD(S) contains 1 and hence contains all constant functions. We complete this proof by showing that $T_zf \in QMD(S)$ for every $z \in MM\ QMD(S)$. Let $z \in MM\ QMD(S)$ and $\theta : x \to MM\ QMD(S)$ be the restriction map. I.e., $\theta(x) = x|_{QMD(S)}$. Then $\theta$ is a continuous homomorphism onto MM QMD(S). There exists $w \in X$ such that $\theta(w) = z$. For $s \in S$, $T_wf(s) = w(L_sf) = z(L_sf) = T_zf(s)$. Thus $T_wf = T_zf$. Let $v \in \overline{X^2}$, $u \in X$, and $e \in E(x)$. Then $veu(T_zf) = veu(T_wf) = veuw(f) = vuw(f)$ (since $f \in QD(S)$) $= vu(T_wf) = vu(T_zf)$. Thus $T_zf \in QD(S)$. Now $ve(T_zf) = ve(T_wf) = vew(f) = vw(f)$ (since $f \in QD(S)$) $= v(T_wf)$. Therefore, by Proposition 2·4, $T_zf \in QMD(S)$.

THEOREM 2·7. QMD(S)-compactification $(Y,\beta)$ of S maximal with respect to the property that $\overline{Y^2}$ is left simple.

PROOF. Let $(X,\alpha)$ denote this LMC(S)-compactification of S and let $\theta: X \to Y$ denote the restriction map. Then $\theta$ is a continuous homomorphism of X onto Y. We first prove that $\overline{Y^2}$ is left simple. Let $u \in Y^2$ and $e \in E(\overline{Y^2})$. There exists $x \in X^2$, $d \in E(X)$ such that $\theta(x) = u$ and $\theta(d) = e$. (We note that $\theta^{-1}(e)$ is a compact subsemigroup of X and hence has an independent d.) For $f \in QMD(S)$, $ue(f) = \theta(x)\theta(d)(f) = \theta(xd)(f) = xd(f) = x(f) = u(f)$. Therefore $ue = u$. Since Y is right topological, $ue = u$ for all $u \in \overline{Y^2}$. Thus $\overline{Y^2}$ is left simple. To prove that $(Y,\beta)$ maximal with respect to this property, we let $(Y_o,\beta_o)$ be a right topological compactification of S with the property that $\overline{Y_o^2}$ is left simple. Then it remains to show that $\beta_o^* C(Y_o) \subseteq QMD(S)$ [1, III Theorem 2·4]. Junghenn has shown that $\beta_o^* C(Y_o) \subseteq LMC(S)$ [2, page 385]. Therefore, there exists a continuous homomorphism $\delta = X \to Y_o$ such that $\delta_o\alpha = \beta_o$. Let $f \in \beta_o^* C(Y_o)$. Then $f = \beta_o^*g$ for some $g \in C(Y_o)$. Now $\alpha(s)(f) = \alpha(s)(\beta_o^*g) = \beta_o^* g(s) = g(\beta_o(s)) = g(\delta(\alpha(s)))$. Since $\alpha$ is dense in X, $x(f) = g(\delta(x))$ for every $x \in X$. Let $u \in \overline{X^2}$, $v \in X$ and $e \in E(X)$. Then $uev(f) = g(\delta(uev)) = g(\delta(u)\delta(e)\delta(v)) = g(\delta(u)\delta(v))$ (since $\overline{Y_o^2}$ is left simple) $= g(\delta(uv)) = uv(f)$. Therefore $f \in QD(S)$. Further $ue(f) = g(\delta(ue)) = g(\delta(u)\delta(e)) = g(\delta(u)) = u(f)$. Thus $f \in QMD(S)$ and the proof is complete.

Remark 2·8. Theorem 2·7 raises a question. Is there an admissible subalgebra $F_n$ of C(S) such that the $F_n$-compactification $(Y,\beta)$ of S is maximal with respect to the property that $\overline{Y^n}$ is left simple? We proved an analogous result for generalized distal functions [3]. We conjecture that the notion of minimal distal functions could be generalized and that the nth order minimal distal function space $MD^n(S)$-compactification $(Y,\beta)$ is maximal with respect to the property that $\overline{Y^{n+1}}$ is left simple.

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