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A CHARACTERIZATION OF THE ROGERS q-HERMITE POLYNOMIALS

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ABSTRACT. In this paper we characterize the Rogers q-Hermite polynomials as the only orthogonal polynomial set which is also $\mathcal{D}_q$-Appell where $\mathcal{D}_q$ is the Askey-Wilson finite difference operator.

KEY WORDS AND PHRASES. Orthogonal polynomials, generating functions, Askey-Wilson operator
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1. INTRODUCTION

Appell polynomials sets $\{P_n(x)\}$ are generated by the relation

$$A(t)e^{xt} = \sum_{n=0}^{\infty} P_n(x)\, t^n, \tag{1.1}$$

where $A(t)$ is a formal power series in t with $A(0) = 1$. This definition implies the equivalent property that

$$DP_n(x) = P_{n-1}(x), \qquad D = d/dx, \tag{1.2}$$

Examples of such polynomial sets are

$$\left\{\frac{x^n}{n!}\right\}, \left\{\frac{B_n(x)}{n!}\right\}, \left\{\frac{H_n(x)}{2^n n!}\right\} \tag{1.3}$$

where $B_n(x)$ is the nth Bernoulli polynomial and $H_n(x)$ is the nth Hermite polynomials generated by

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}. \tag{1.4}$$

By an orthogonal polynomial set (OPS) we shall mean those polynomial sets which satisfy a three term recurrence relation of the form

$$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \qquad (n = 0, 1, 2, \cdots) \tag{1.5}$$

with $P_0(x) = 1,\ P_{-1}(x) = 0$, and $A_n A_{n-1} C_n > 0$.

By Favard's theorem [7] this is equivalent to the existence of a positive measure $d\alpha(x)$ such that

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)\, d\alpha(x) = K_n \delta_{nm}. \tag{1.6}$$

As we see from the examples (1.3) some Appell polynomials are orthogonal and some are not. This prompted Angelesco [3] to prove that *the only orthogonal polynomial sets which are also Appell is the Hermite polynomial set.* This theorem was rediscovered by several authors later on (see, e.g., [10]).

There were several extensions and/or analogs of Appell polynomials that were introduced later. Some are based on changing the operator D in (1.2) into another differentiation-like operator or by replacing the generating relation (1.1) by a more general one. In most of these cases theorems like Angelesco's were given. For example Carlitz [6] proved that the Charlier polynomials are the only OPS which satisfy the difference relation

$$\Delta P_n(x) = P_{n-1}(x), \qquad (\Delta f(x) = f(x+1) - f(x).) \tag{1.7}$$

See [1] for many other references.

A new and very interesting analog of Appell polynomials were introduced recently, as a biproduct of other considerations, by Ismail and Zhang [9]. In discussing the Askey-Wilson operator they defined a new q-analog of the exponential function e^{xt}. This we describe in the next section.

2. NOTATIONS AND DEFINITIONS

The Askey-Wilson operator is defined by

$$\mathcal{D}_q f(x) = \frac{\delta_q f(x)}{\delta_q x}, \tag{2.1}$$

where $x = \cos\theta$ and

$$\delta_q g(e^{i\theta}) = g(q^{1/2}e^{i\theta}) - g(q^{-1/2}e^{i\theta}). \tag{2.2}$$

We further assume that $-1 < q < 1$ and use the notation

$$(a;q)_0 = 1, \qquad (a;q)_n = (1-a)(1-qa)\cdots(1-aq^{n-1}), \quad (n = 1,2,..) \tag{2.3}$$

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^j). \tag{2.4}$$

There are two q-analogs of the exponential function e^x given by the infinite products

$$e_q(x) = \frac{1}{(x;q)_\infty} = \sum_{k=0}^{\infty}\frac{x^k}{(q;q)_k}, \tag{2.5}$$

and

$$\frac{1}{e_q(x)} = (x;q)_\infty = \sum_{k=0}^{\infty}(-1)^k\frac{q^{\frac{1}{2}k(k-1)}}{(q;q)_k}x^k. \tag{2.6}$$

We shall also use the function

$$\Psi_n(x) = i^n(iq^{(1-n)/2}e^{i\theta};q)_n(iq^{(1-n)/2}e^{-i\theta};q)_n, \tag{2.7}$$

so that

$$\begin{aligned}
\Psi_{2n}(x) &= \prod_{k=0}^{n-1}\left[4x^2 + (1-q^{2n-1-2k})(1-q^{1-2n+2k})\right] \\
\Psi_{2n+1}(x) &= 2x\prod_{k=0}^{n-1}\left[4x^2 - (1-q^{2n-2k})(1-q^{-2n+2k})\right] \\
4x^2\Psi_n(x) &= \Psi_{n+2}(x) + (1-q^{n+1})(1-q^{-n-1})\Psi_n(x)
\end{aligned} \tag{2.8}$$

Thus

$$\mathcal{D}_q \Psi_n(x) = 2q^{(1-n)/2}\frac{1-q^n}{1-q}\ \Psi_{n-1}(x). \tag{2.9}$$

and

$$\mathcal{D}_q\left[x\ \Psi_n(x)\right] = \frac{q^{(1+n)/2} - q^{-(n+1)/2}}{q^{1/2} - q^{-1/2}}\ 2x\ \Psi_{n-1}(x). \tag{2.10}$$

Iterating (2.9) we get

$$\mathcal{D}_q^k \Psi_n(x) = \ 2^k q^{\frac{1}{4}k(k+1)-\frac{1}{2}nk}\frac{(q;q)_n}{(q;q)_{n-k}(1-q)^k}\Psi_{n-k}(x). \tag{2.11}$$

The Ismail-Zhang q-analog of the exponential function [9] is

$$\mathcal{E}(x) = \sum_{n=0}^{\infty}\frac{q^{n(n-1)/4}(1-q)^n}{2^n(q;q)_n}\ \Psi_n(x)\ t^n. \tag{2.12}$$

It follows from (2.12) and (2.9) that

$$\mathcal{D}_q\mathcal{E}(x) = t\ \mathcal{E}(x). \tag{2.13}$$

This suggested to Ismail and Zhang to define the $\mathcal{D}_q$-Appell polynomials as those, in analogy with (1.1), defined by

$$A(t)\mathcal{E}(x) = \sum_{n=0}^{\infty} P_n(x)\ t^n, \tag{2.14}$$

so that

$$\mathcal{D}_q P_n(x) = P_{n-1}(x). \tag{2.15}$$

An example of such a set is the Rogers q-Hermite polynomials, $\{H_n(x|q)\}$, (see [2, 4, 8]).

$$\prod_{n=0}^{\infty}\left(1 - 2xtq^n + t^2q^{2n}\right)^{-1} = \sum_{n=0}^{\infty} H_n(x|q)\frac{t^n}{(q;q)_n}. \tag{2.16}$$

They satisfy the three term recurrence relation

$$H_{n+1}(x|q) = 2xH_n(x|q) - (1-q^n)H_{n-1}(x|q), \qquad n = 0,1,2,3,... \tag{2.17}$$

with $H_0(x|q) = 1,\ H_{-1}(x|q) = 0$.

3. THE MAIN RESULT

We now state our main result:

Theorem 1. *The orthogonal polynomial sets which are also $\mathcal{D}_q$-Appell, i.e., satisfy (2.15) or (2.14) is the set of the Rogers q-Hermite polynomials.*

Proof Let $\{Q_n(x)\}$ be a polynomial set which is both orthogonal and $\mathcal{D}_q$-Appell. That is $\{Q_n(x)\}$ satisfy (2.14) and (1.5).

We next note that (2.16) implies that

$$h_n(x|q) = \frac{(1-q)^n q^{n(n-1)/4}}{2^n(q;q)_n}H_n(x|q) \tag{3.1}$$

satisfy

$$\mathcal{D}_q h_n(x|q) = h_{n-1}(x|q), \tag{3.2}$$

so that $\{h_n(x|q)\}$ is a $\mathcal{D}_q$–Appell polynomial set and at the same time is an OPS satisfying the three term recurrence relation

$$(1-q^{n+1})h_{n+1}(x|q) = (1-q)q^{n/2}xh_n(x|q) - \frac{1}{4}(1-q)^2q^{n-1/2}h_{n-1}(x|q) \tag{3.3}$$

It also follows from (2.14) that any two polynomial sets $\{R_n(x)\}$ and $\{S_n(x)\}$, in that class are related by $R_n(x) = \sum_{k=0}^{n} c_{n-k}S_k(x)$. Thus the solution to our problem may be expressed as

$$Q_n(x) = \sum_{k=0}^{n} a_{n-k}h_k(x|q). \tag{3.4}$$

for some sequence of real constants $\{a_n\}$. We may assume without loss of generality that $a_0 = 1$.

The three term recurrence relation satisfied by $\{Q_n(x)\}$ is

$$(1-q^{n+1})Q_{n+1}(x) = \left((1-q)q^{n/2}x + \beta_n\right)Q_n(x) - \gamma_nQ_{n-1}(x), \tag{3.5}$$

with $Q_0(x) = 1,\ Q_{-1}(x) = 0$. Thus $Q_1(x) = x + \beta_0 = a_1 + h_1(x|q)$, from which it follows that $a_1 = \beta_0$.

Putting (3.4) in (3.5) and using (3.3) to replace $xh_k(x|q)$ in terms of $h_{k+1}(x|q)$ and $h_{k-1}(x|q)$ we get, on equating coefficients of $h_k(x|q)$,

$$(1-q^{(n-k+1)/2})(1+q^{(n+1+k)/2})a_{n+1-k} - \beta_na_{n-k} + \left[\gamma_n - \frac{1}{4}(1-q)^2q^{(n+k)/2}\right]a_{n-k-1} = 0, \tag{3.6}$$

valid for all n and $k = 0,1,2,...,n+1$ provided we interpret $a_{-1} = a_{-2} = 0$. It is easy to see that this system of equations is equivalent to the solution of our problem.

Putting $k = n$ in (3.6) we get

$$\beta_n = (1-q^{\frac{1}{2}})(1+q^{n+\frac{1}{2}})a_1. \tag{3.7}$$

Hence if $\beta_0 = 0$ then $\beta_n = 0$ for all n. In fact if $\beta_m = 0$ for any $n = m$ then $\beta_n = 0$ for all n.

Now we treat these two cases seperately.

Case I. ($\beta_0 = 0$).

The system (3.6) can now be written as

$$(1-q^{(k+1)/2})(1+q^{n+\frac{1}{2}(1-k)})a_{k+1} + \left[\gamma_n - \frac{1}{4}(1-q)^2q^{n-\frac{1}{2}k}\right]a_{k-1} = 0. \tag{3.8}$$

Since $a_1 = 0$ then it follows from (3.8) that $a_{2k+1} = 0$ for all k. In particular we get

$$\gamma_n = \frac{1}{4}(1-q)^2q^{n-\frac{1}{2}} - a_2(1-q)(1+q^n), \tag{3.9}$$

so that if $a_2 = 0$ then

$$Q_n(x) = h_n(x|q). \tag{3.10}$$

Now we show that $a_2 \neq 0$ leads to contradiction. To do this replace k by $2k-1$. We get

$$(1-q^k)(1+q^{n-k+1})a_{2k} + \left[\frac{1}{4}(1-q)^2q^{n-\frac{1}{2}}(1-q^{1-k}) - a_2(1-q)(1+q^n)\right]a_{2k-2} = 0. \tag{3.11}$$

Keep k fixed and let $n \to \infty$. We get $(1-q^k)a_{2k} = (1-q)a_2a_{2k-2}$. Thus

$$a_{2k} = \frac{(1-q)^k}{(q;q)_k}a_2^k. \tag{3.12}$$

Putting this value in (3.11) we get $q^{1-k} = 1$. This is a contradiction and Case I is finished.

Case II ($\beta_0 \neq 0$).

We start with (3.6) we get, assuming $a_1 \neq 0$,

$$\gamma_n = \frac{1}{4}(1-q)^2 q^{n-\frac{1}{2}} + (1-q^{\frac{1}{2}})(1+q^{n+\frac{1}{2}})a_1^2 - (1-q)(1+q^n)a_2. \tag{3.13}$$

Putting this value of γ_n and the value of β_n in (3.7) in (3.6), and finally equating coefficients of q^n and the terms independent of n we get the pair of equation systems

$$(1-q^{(k+1)/2})a_{k+1} - (1-q^{\frac{1}{2}})a_1 a_k + \left\{(1-q^{\frac{1}{2}})a_1^2 - (1-q)a_2\right\} a_{k-1} = 0 \tag{3.14}$$

and

$$\begin{aligned}(1-q^{(k+1)/2})a_{k+1} - (1-q^{\frac{1}{2}})q^{k/2}a_1 a_k +& \\ \left\{\frac{1}{4}(1-q)^2 q^{-\frac{1}{2}}(q^{(k-1)/2}-1) + q^{k/2}(1-q^{\frac{1}{2}})a_1^2 - (1-q)q^{(k-1)/2}a_2\right\} a_{k-1} = 0&\end{aligned} \tag{3.15}$$

Eliminating a_{k+1} in these equations we get

$$\begin{aligned}(1-q^{\frac{1}{2}})(1-q^{k/2})a_1 a_k + \Big\{(1-q)a_2(1-q^{(k-1)/2})& \\ - (1-q^{\frac{1}{2}})(1-q^{k/2})a_1^2 - \frac{1}{4}(1-q)^2 q^{-\frac{1}{2}}(1-q^{(k-1)/2})\Big\} a_{k-1} = 0.&\end{aligned} \tag{3.16}$$

This equation is of the form $(1-q^{k/2})a_1 a_k = c(1-bq^{k/2})a_{k-1}$ so that the general solution of (3.16) is

$$a_k = c^k \frac{(bq^{\frac{1}{2}}; q^{\frac{1}{2}})_k}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_k} \tag{3.17}$$

Putting this in (3.14) we get that $b = 0$. On the other hand (3.15) gives that $c^2 = \frac{1}{4}(1-q)^2 q^{-\frac{1}{2}}$. Finally putting those values of a_k in (3.13) we get that $\gamma_n = 0$ which is a contradiction.

This completes the proof of the theorem.

4. GENERATING FUNCTION

We obtain, for the q-Hermite polynomials, a generating function of the form (2.14). More specifically we prove

Theorem 2. *Let $H_n(x|q)$ be the nth Rogers q-Hermite polynomial. Then we have*

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/4}}{(q;q)_n} H_n(x|q)\, t^n = (t^2 q^{-\frac{1}{2}}; q^2)_\infty \mathcal{E}(x). \tag{4.1}$$

Proof. Let $A(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$ and

$$A(t)\mathcal{E}(x) = \sum_{n=0}^{\infty} h_n(x|q) t^n. \tag{4.2}$$

Then we get

$$h_n(x|q) = \sum_{k=0}^{n} a_{n-k} c_k \Psi_k(x). \tag{4.3}$$

where

$$c_k = \frac{(1-q)^k}{2^k (q;q)_k} q^{k(k-1)/4}. \tag{4.4}$$

To calculate the coefficients $\{a_n\}$ we first iterate (3.3) we get

$$
\begin{aligned}
4x^2h_n(x|q) &= \frac{4}{(1-q)^2}(1-q^{n+1})(1-q^{n+2})q^{-n-\frac{1}{2}}h_{n+2}(x|q) \\
&+ (2-q^n-q^{n+1})h_n(x|q)+\frac{(1-q)^2}{4}q^{n-\frac{3}{2}}h_{n-2}(x|q).
\end{aligned}
\tag{4.5}
$$

Putting (4.3) in (4.5), using (2.6) and then equating coefficients of $\Psi_k(x)$ we get after some simplification

$$
\begin{aligned}
&\frac{4}{(1-q)^2}q^{-n-\frac{1}{2}}(1-q^{n-k+2})(1-q^{n+k+1})a_{n+2-k}+ \\
&q^{-k-1}\left\{1+q^{2k+2}-q^{n+k+1}-q^{n+k+2}\right\}a_{n-k}+ \\
&\frac{(1-q)^2}{4}q^{n-\frac{3}{2}}a_{n-2-k}=0 \qquad (k=0,1,...,n+2).
\end{aligned}
\tag{4.6}
$$

By direct calculation of a_1, a_2, a_3 we see easily that $a_1=a_3=0$. Thus (4.6) shows that $a_{2k+1}=0$ for all k.

Furthermore we can easily verify that

$$
a_{2j}=(-1)^j\frac{(1-q)^{2j}}{2^{2j}(q^2;q^2)_j}\,q^{j(j-\frac{3}{2})} \qquad (j=0,1,2,3,... \tag{4.7}
$$

Hence

$$
\begin{aligned}
A(t) &= \sum_{j=0}^{\infty}(-1)^j\frac{q^{j(j-1)}}{(q^2;q^2)_j}\left(\frac{(1-q)^2t^2}{4}q^{-\frac{1}{2}}\right)^j \\
&= \left(\frac{(1-q)^2}{4}t^2q^{-\frac{1}{2}};q^2\right)_\infty .
\end{aligned}
\tag{4.8}
$$

After some rescaling we get the theorem.

As a corollary of (4.1) we state the pair of inverse relations

$$
\Psi_n(x) = \sum_k\frac{(q;q)_nq^{k(k-n)}}{(q^2;q^2)_k(q;q)_{n-2k}}H_{n-2k}(x|q), \tag{4.9}
$$

$$
H_n(x|q) = \sum_k(-1)^k\frac{(q;q)_n\,q^{k(2k-n-1)}}{(q^2;q^2)_k(q;q)_{n-2k}}\Psi_{n-2k}(x). \tag{4.10}
$$

These follows from the identities (2.5) and (2.6)

Formula (4.10) and (2.11) give

$$
H_n(x|q) = \frac{1}{e_{q^2}(\frac{(1-q)^2}{4}q^{-\frac{1}{2}}\mathcal{D}_q^2)}\Psi_n(x). \tag{4.11}
$$

This is a q-analog of the formula

$$
e^{-D^2}x^n=H_n(x)
$$

for the regular Hermite polynomials (1.4).

References

[1] W. A. Al-Salam, *Characterization theorems for orthogonal polynomials*, in "Orthogonal Polynomials: Theory and Practice", P. Nevai ed., Kluwer, Dordrecht, 1989, pp. 1–24.

[2] W. A. Al-Salam and M. E. H. Ismail, q-beta integrals and the q-Hermite polynomials, Pacific J. of Math., **135**(1988), 209–221.

[3] A. Angelesco, Sur les polynomes orthogonaux en rapport avec d'autre polynomes, Buletinul Societâtii din Cluj, **1**(1921), 44–59.

[4] R. Askey and M. E. H. Ismail, *A generalization of the ultraspherical polynomials*, Studies in Pure Mathematics, edited by P. Erdős, pp. 55-78, Birkhüser, Basel, 1983.

[5] R. Askey and J. Wilson, *Some basic hypergeometric polynomials that generalize Jacobi polynomials* , Memoirs Amer. Math. Soc. #319, Providence, 1985.

[6] L. Carlitz, *Characterization of certain sequences of orthogonal polynomials*, Portugaliae Math, **20**(1961), 43–46.

[7] T. S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, 1978.

[8] M. E. H Ismail and D. Stanton, *On the Askey-Wilson and Rogers polynomials*, Canad. J. Math. **40** (1988), 1025–1045.

[9] M. E. H. Ismail and R. Zhang, *Diagonalization of certain integral operators*, Advances in Mathematics, to appear.

[10] J. Shohat, *The relation of the classical orthogonal polynomials to the polynomials of Appell*, American J. Math., **58**(1936), 453-464.

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