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POINT-VALUED MAPPINGS OF SETS

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ABSTRACT. Let X be a metric space and let $CB(X)$ denote the closed bounded subsets of X with the Hausdorff metric. Given a complete subspace Y of $CB(X)$, two fixed point theorems, analogues of results in [1], are proved, and examples are given to suggest their applicability in practice.

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Let X be a metric space with metric d and let Y be a complete subspace of the space $CB(X)$ of all closed and bounded subsets of X, with the Hausdorff metric ρ:

$$\rho(A,B) = \max\{\sup_{x\in B} d(x,A), \sup_{x\in A} d(x,B)\}. \tag{1}$$

In Hicks [1], fixed point theorems for set-valued maps $T : X \to CB(X)$ were proved; and illustrated with examples. We show that similar results for maps $T : Y \to X$ can be obtained, using essentially the same techniques as in Hicks [1].

THEOREM 1. Let $T : Y \to X$ be continuous. Then there is an $A \in Y$ such that $T(A) \in A$ iff there exists a sequence $\{A_n\}_{n=0}^{\infty}$ in Y with $T(A_n) \in A_{n+1}$ (or $T(A_{n+1}) \in A_n$) and

$$\sum_{n=0}^{\infty} \rho(A_n, A_{n+1}) < \infty. \tag{2}$$

In this case, $A_n \to A$ as $n \to \infty$. (In fact, we may let $A_{n+1} = A_n \cup \{T(A_n)\}$, for each n, for the case $T(A_n) \in A_{n+1}$.)

PROOF. If $T(A) \in A$, then we are done. Conversely, if the given conditions are met, then $\{A_n\}_{n=0}^{\infty}$ is Cauchy, so let $A \in Y$ be its limit. Thus $T(A_n) \to T(A)$. If $y \in A$, then

$$d(y, T(A)) \le d(y, T(A_n)) + d(T(A_n), T(A)), \tag{3}$$

so

$$d(A, T(A)) \le d(A, T(A_n)) + d(T(A_n), T(A)). \tag{4}$$

Since $d(,T(A_n),T(A)) \to 0$ and we have $d(A,T(A_n)) \leq \rho(A,A_{n+1}) \to 0$, it follows that $T(A) \in A$. ■

EXAMPLES

(1) Let $X = \mathbb{R}$, with the usual metric. Define $T: CB(\mathbb{R}) \to \mathbb{R}$ by

$$T(A) = \alpha \sup(A) + (1-\alpha)\inf(A), \tag{5}$$

where $\alpha \in [0,1]$. Then T is continuous. If $A \in CB(\mathbb{R})$, then

$$T(A \cup \{T(A)\}) = T(A) \in A \cup \{T(A)\}. \tag{6}$$

(2) Let $X = \mathbb{R}$ as in 1, and let $r: [0,\infty) \to [0,\infty)$ be such that $r \sim 1_{\mathbf{R}}$, where $1_{\mathbf{R}}$ is the identity on $\mathbb{R}$. Define $T: CB(\mathbb{R}) \to \mathbb{R}$ by

$$T(A) = \alpha r(|\sup(A)|) + (1-\alpha) r(|\inf(A)|), \tag{7}$$

where $\alpha \in (0,1)$. Assuming r is continuous, so is T. Let $A_0 \in CB(\mathbb{R})$, and for $n \in \mathbb{N}$, let

$$A_{n+1} = A_n \cup \left[\inf_{k \leq n}\{T(A_n)\}, \sup_{k \leq n}\{T(A_k)\}\right]. \tag{8}$$

Theorem 1 yields $A \in CB(\mathbb{R})$ with $T(A) \in A$ if

$$\sum_{n=1}^{\infty} \max\left\{d\left(\inf_{k \leq n}\{T(A_n)\}, A_n\right), d\left(\sup_{k \leq n}\{T(A_k)\}, A_n\right)\right\} < \infty. \tag{9}$$

DEFINITION. Let (X,d) be a metric space and let Y be a subspace of $(CB(X),\rho)$. Let $T: Y \to X$. Then T is <u>nice</u> if for each $A \in Y$ and each $x \in A$ with $d(x,T(A)) = d(A,T(A))$, there exists a set $B \in Y$ with $T(B) = x$.

EXAMPLES

(3) Let $X = \mathbb{R}^2$, $T: CB(\mathbb{R}^2) \to \mathbb{R}^2$ defined by

$$T(A) = (\inf(\mathrm{proj}_1(A)), \sup(\mathrm{proj}_1(A))). \tag{10}$$

Let $a > b$ and $A = [0,a] \times [0,b]$. Then $T(A) = (0,a)$, and $(0,b)$ is the only point of A whose distance from $(0,a)$ equals $d(A,T(A))$. Let $B = [0,b]^2$. Then $T(B) = (0,b)$.

(4) Let $X = \mathbb{R}^2$, and for $A \in CB(\mathbb{R}^2)$, let $T(A)$ be the center of the circle which circumscribes A. Let $r = d(A,T(A))$, and let $x \in A$ with $d(x,T(A)) = r$. Let $B = A \cap \overline{\mathscr{B}\left(x, \frac{diam(A)}{2}\right)}$. Then $T(B) = x$.

THEOREM 2. Let (X,d) be a metric space and let Y be a complete subspace of $(CB(X),\rho)$, each member of which is compact. Let $T: Y \to X$ be continuous. Assume that $K: [0,\infty) \to [0,\infty)$ is non-decreasing, $K(0) = 0$, and

$$\rho(A,B) \leq K(d(T(A),T(B))) \tag{11}$$

for A, $B \in Y$. If T is nice, then there is $A \in Y$ such that $T(A) \in A$ iff there exists $A_0 \in Y$ for which

$$\sum_{n=1}^{\infty} K^n\big(d(A_0, T(A_0))\big) < \infty \tag{*}$$

In this case, we can choose $\{A_n\}_{n=1}^{\infty}$ such that $T(A_{n+1}) \in A_n$ and $A_n \to A$.

PROOF. If $T(A) \in A$, then we are done. If $A_0 \in Y$ satisfies (*), let $x_1 \in A_0$ with $d(x_1, T(A_0)) = d(A_0, T(A_0))$. Since T is nice, let $A_1 \in Y$ with $T(A_1) = x_1$.

Next, let $x_2 \in A_1$ with $d(x_2, T(A_1)) = d(A_1, T(A_1))$, and then let $A_2 \in Y$ with $T(A_2) = x_2$. Then

$$\begin{aligned} d(T(A_1), T(A_2)) &= d(T(A_1), x_2) \\ &= d(T(A_1), A_1) = d(x_1, A_1) \\ &\le \rho(A_0, A_1) \le K\big(d(T(A_0), T(A_1))\big), \end{aligned} \tag{12}$$

so that

$$\begin{aligned} K\big(d(T(A_1), T(A_2))\big) &\le K^2\big(d(T(A_0), T(A_1))\big) \\ &= K^2\big(d(T(A_0), x_1)\big) \\ &= K^2\big(d(T(A_0), A_0)\big). \end{aligned} \tag{13}$$

Now, suppose we have $x_n \in A_{n-1}$ and $A_n \in Y$ with $d(x_n, T(A_{n-1})) = d(A_{n-1}, T(A_{n-1}))$ and $T(A_n) = x_n$. Let $x_{n+1} \in A_n$ with $d(x_{n+1}, T(A_n)) = d(A_n, T(A_n))$ and let $A_{n+1} \in Y$ with $T(A_{n+1}) = x_{n+1}$. Then

$$\begin{aligned} d(T(A_n), T(A_{n+1})) &= d(T(A_n), x_{n+2}) \\ &= d(T(A_n), A_n) = d(x_n, A_n) \\ &\le \rho(A_{n-1}, A_n) \le K\big(d(T(A_{n+1}), T(A_n))\big), \end{aligned} \tag{14}$$

so that

$$\begin{aligned} K\big(d(T(A_n), T(A_{n+1}))\big) &\le K^2\big(d(T(A_{n-1}), T(A_n))\big) \\ &= K\Big(K\big(d(T(A_{n-1}), T(A_n))\big)\Big) \\ &\le K\Big(K^2\big(d(T(A_{n-2}), T(A_{n-1}))\big)\Big) \\ &= K^3\big(d(T(A_{n-2}), T(A_{n-1}))\big) \\ &\le \cdots \le K^n\big(d(T(A_0), A_0)\big). \end{aligned} \tag{15}$$

Thus, since

$$\rho(A_n, A_{n+1}) \le K\big(d(T(A_n), T(A_{n+1}))\big), \tag{16}$$

it follows from (*) that

$$\sum_{n=0}^{\infty} \rho(A_n, A_{n+1}) < \infty, \tag{17}$$

and then by Theorem 1, $A_n \to A$ and $T(A) \in A$. ■

Note that the conditions of theorem 2 force T to be a bijection. In both of these theorems, we have used completeness of the given subspace Y of $CB(X)$ instead of completeness of X. In fact, in theorem 2, since T is a bijection, we may trade completeness of Y back for completeness of X and use the second theorem of Hicks [1].

THEOREM 3. If (X,d) is a complete metric space and Y is any subspace of $(CB(X),\rho)$, each member of which is compact, then for any homeomorphism $T: Y \to X$ such that

$$\rho(A,B) \le K\big(d(T(A),T(B))\big), \tag{18}$$

where $K:[0,\infty)\to[0,\infty)$ is nondecreasing, with $K(0)=0$, there is $A \in Y$ such that $T(A) \in A$ iff there exists $A_0 \in Y$ for which (*) holds.

PROOF. If $A_0 \in Y$ satisfies (*), let $x_0 = T(A_0)$. Apply theorem 2 of Hicks [1] to $T^{-1}: X \to Y$ to obtain a $p \in X$ such that $p \in T^{-1}(p)$. Let $A = T^{-1}(p)$. Then $T(A) = p$ is in A, so we are done. ■

REFERENCES

[1] HICKS, T.L. Fixed-Point Theorems for Multivalued Mappings, Indian J. Pure Appl. Math. 20(11) (November 1989), 1077-1079