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FURTHER RESULTS ON A GENERALIZATION OF BERTRAND'S POSTULATE

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ABSTRACT. Let d(k) be defined as the least positive integer n for which $p_{n+1} < 2p_n - k$. In this paper we will show that for $k \geq 286664$, then $d(k) < k/(\log k - 2.531)$ and for $k \geq 2$, then $k(1-1/\log k)/\log k < d(k)$. Furthermore, for k sufficiently large we establish upper and lower bounds for d(k).

KEY WORDS AND PHRASES. Bertrand's Postulate, Primes.
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1. INTRODUCTION.

Let p_n be the n^{th} prime and let k be a positive integer. We define d(k) to be the least positive integer n for which $p_{n+1} < 2p_n - k$, and consider the corresponding generalization of Bertrand's Postulate. There are other generalizations of Bertrand's Postulate, for example [1] and [2].

Dressler [3] showed that $p_{n+1} < 2p_n - 10$ for all $n > 6$. Badea [4] proved that for every integer $k \geq 1$ we have

$$d(k) \leq (M_k + 2 + ((M_k)^2 + 12\, M_k + 4)^{1/2})/4$$

where $M_k = \max\ (118,\ [13\ k/12] + 1)$. The Mathematical Review (88j:11005) of [4] points out two facts. First, "Since the prime number theorem implies $p_{n+1} \sim p_n \sim n\log n$ it follows that $d(k) \sim k/\log k$." Second, the reviewer states that using the results found in [4] he can establish, using an elementary argument, the following:

$$d(k) \leq [13\ k/(12(\log k - \log\log k))] + 1 \quad \text{for every } k \geq 4.$$

In this paper we will use improved upper and lower bounds for p_n and thereby establish an improved upper bound for d(k). Furthermore, we will give an explicit lower bound for d(k). It is obvious that in order to establish an upper bound for d(k) we would want to find conditions on n which guarantee

$$k < 2p_n - p_{n+1}. \tag{1.1}$$

Also in order to establish a lower bound for d(k) we would want to find conditions on n which guarantee

$$k > 2p_n - p_{n+1}. \tag{1.2}$$

To obtain an explicit upper bound for d(k) we have to observe that $k/(\log k - \log\log k) < k/(\log k - 2.531)$ if $k \leq 286663$. Hence we have different upper bound functions for d(k) depending upon the value of k. Moreover, we need to use the computer languages Maple and Turbo Pascal. We use Maple to get the exact value of k which is needed in Lemma 7, and we use Turbo Pascal to obtain Tables 1 and 2 for small values of k and provide a program to verify Cases 1-3 of Theorem 4.

The proofs of this paper require the following results.

$$n(\log n + \log\log n - 3/2) < p_n \qquad n \geq 2 \tag{1.3}$$

$$p_n < n(\log n + \log\log n - 1/2) \qquad n \geq 20 \tag{1.4}$$

$$p_n = n(\log n + \log\log n - 1 + O(\log\log n/\log n)) \tag{1.5}$$

$$p_{n+1} - p_n \leq 652 \qquad p_n \leq 2.686 \cdot 10^{12} \tag{1.6}$$

$$\log\log(n+1) < .000003412 + \log\log n \qquad n \geq 28567 \tag{1.7}$$

$$n\log(1+1/n)+\log(n+1)+\log\log(n+1) < .00053582n \quad n \geq 28567 \tag{1.8}$$

(1.3) and (1.4) are found in [5], (1.5) is found in [6] and (1.6) is found in [7].

We define the following function to make Lemmas 2 and 4 more readable,

$$T(k,c) = (1+c\varepsilon)/\log k \qquad \text{where } \varepsilon > 0. \tag{1.9}$$

2. THEOREMS, LEMMAS AND THEIR PROOFS.

LEMMA 1. For n sufficiently large, there exists $\varepsilon > 0$ such that $2p_n - p_{n+1} > n(\log n + \log\log n - (1 + \varepsilon))$.

PROOF. From (1.5) there exists a constant c such that $p_n < n(\log n + \log\log n - 1 + c(\log\log n / \log n)$ and

$p_n > n(\log n + \log\log n - 1 - c(\log\log n / \log n)$.

We see that

$$2p_n - p_{n+1} > 2n(\log n + \log\log n - 1 - c(\log\log n/\log n)) - (n+1)\{\log(n+1) + \log\log(n+1) - 1 + c(\log\log(n+1)/\log(n+1))\}. \quad (2.1)$$

After simplification and for n sufficiently large (2.1) will become $2p_n - p_{n+1} > n(\log n + \log\log n - (1+\varepsilon))$. QED.

LEMMA 2. With ε and n as in Lemma 1, let $n = k(1+(1+4\varepsilon)/\log k)/\log k$ then for k sufficiently large we have $k < n(\log n + \log\log n - (1+\varepsilon))$.

PROOF. Suppose not; then

$$k \geq n(\log n + \log\log n - (1+\varepsilon)). \quad (2.2)$$

After substituting for n, multiplying through by $(\log k)/k$ and using (1.9), (2.2) becomes

$$\begin{aligned}\log k > \log k &+ \log(1+T(k,4)) - \log\log k - (1+\varepsilon) \\ &+ \log\log((k/\log k)(1+T(k,4))) + T(k,4)\log k \\ &+ T(k,4)\log(1+T(k,4)) - (1+\varepsilon)T(k,4) \\ &- T(k,4)\{\log\log k - \log\log((k/\log k)(1+T(k,4)))\}. \quad (2.3)\end{aligned}$$

We observe that (2.3) does not hold for large k because $-\log\log k + \log\log((k/\log k)(1+T(k,4))) \to 0$ and the $T(k,4)\log k$ term dominates. Hence this establishes the Lemma. QED.

THEOREM 1. With ε as in Lemma 1, there exists k sufficiently large such that $d(k) < k(1+(1+4\varepsilon)/\log k)/\log k$.

PROOF. We want to find an upper bound for the function d(k) such that for all $n \geq d(k)$ we have

$$k < 2p_n - p_{n+1}. \quad (2.4)$$

For n sufficiently large and $\varepsilon > 0$, by Lemma 1 we have the following inequality

$$n(\log n + \log\log n - (1+\varepsilon)) < 2p_n - p_{n+1}. \quad (2.5)$$

From (2.5) we now replace (2.4) with a more restrictive inequality

$$k < n(\log n + \log\log n - (1+\varepsilon)). \quad (2.6)$$

Choose $n = k(1+(1+4\varepsilon)/\log k)$. Then by Lemma 2, (2.6) and hence (2.4) still hold. Therefore $d(k) < n=k(1+(1+4\varepsilon)/\log k)/\log k$, establishing an upper bound for $d(k)$. QED.

LEMMA 3. For n sufficiently large, there exists $\varepsilon > 0$ such that $2p_n - p_{n+1} < n(\log n + \log\log n - (1-\varepsilon))$.

PROOF. By using the upper and lower bounds for p_n found in the proof of Lemma 1, we have the following

$$\begin{aligned}2p_n - p_{n+1} < 2n(\log n + \log\log n - 1 + c(\log\log n/\log n)) \\ - (n+1)\{\log(n+1) + \log\log(n+1) - 1 \\ - c(\log\log(n+1)/\log(n+1))\}. \qquad (2.7)\end{aligned}$$

After simplification of (2.7) we have for n sufficiently large the desired result $2p_n - p_{n+1} < n(\log n + \log\log n - (1-\varepsilon))$. QED.

LEMMA 4. With ε and n as in Lemma 3, let $n = k(1+(1-3\varepsilon)/\log k)/\log k$ then for k sufficiently large we have $k > n(\log n + \log\log n - (1-\varepsilon))$.

PROOF. Suppose not; then

$$k \leq n(\log n + \log\log n - (1-\varepsilon)). \qquad (2.8)$$

After substituting for n, multiplying through by $(\log k)/k$ and using (1.9), (2.8) becomes

$$\begin{aligned}\log k \leq \log k + \log(1+T(k,-3)) - \log\log k - (1-\varepsilon)T(k,-3) \\ + \log\log((k/\log k)(1+T(k,-3))) + T(k,-3)\log k \\ + T(k,-3)\log(1+T(k,-3)) - T(k,-3)\log\log k - (1-\varepsilon) \\ + T(k,-3)\log\log((k/\log k)(1+T(k,-3))). \qquad (2.9)\end{aligned}$$

We observe that (2.9) does not hold for large k because $-\log\log k + \log\log((k/\log k)(1+T(k,-3))) \rightarrow 0$ and the $T(k,-3)\log k$ term dominates, thereby proving the Lemma. QED.

THEOREM 2. With ε as in Lemma 3, there exists k sufficiently large such that $d(k) > k(1+(1-3\varepsilon)/\log k)/\log k$.

PROOF. We want to find a lower bound for $d(k)$ such that for all $n < d(k)$ we have

$$k > 2p_n - p_{n+1}. \qquad (2.10)$$

For n sufficiently large, Lemma 3 yields the following inequality

$$n(\log n + \log\log n - (1-\varepsilon)) > 2p_n - p_{n+1}. \qquad (2.11)$$

From (2.11) we now replace (2.10) with a more restrictive inequality

$$k > n(\log n + \log\log n - (1-\varepsilon)). \qquad (2.12)$$

Choose $n = k(1+(1-3\varepsilon)/\log k)/\log k$. Then by Lemma 4, (2.12) and hence (2.10) still hold. Therefore
$d(k) > n = k(1+(1-3\varepsilon)/\log k)/\log k$ establishing a lower bound for $d(k)$. QED.

LEMMA 5. If $n \geq 20$, then
$2p_n - p_{n+1} < n(\log n + \log\log n + 1/2)$.

PROOF. From (1.3) and (1.4) we have

$$2p_n - p_{n+1} < 2n(\log n + \log\log n - 1/2) - (n+1)(\log(n+1) + \log\log(n+1) - 3/2). \qquad (2.13)$$

After several manipulations we see that (2.13) becomes

$$2p_n - p_{n+1} < n(\log n + \log\log n + 1/2). \qquad \text{QED.}$$

LEMMA 6. Let $n = k(1 - 1/\log k)/\log k$, where $k \geq 92$ then $k > n(\log n + \log\log n + 1/2)$.

PROOF. Suppose not; then

$$k \leq n(\log n + \log\log n + 1/2). \qquad (2.14)$$

After substituting for n and multiplying through by $(\log k)/k$, (2.14) would become

$$\begin{aligned}\log k \leq{} & \log k - \log\log k + \log(1 - 1/\log k) \\ & + \log\log(k(1 - 1/\log k)/\log k) + 1/2 \\ & + \{\log\log k - \log k - \log(1 - 1/\log k) \\ & \quad - \log\log(k(1 - 1/\log k)/\log k) - 1/2\}/\log k. \qquad (2.15)\end{aligned}$$

With further simplifications and rearrangement of terms we see that (2.15) is false for $k \geq 92$, thereby establishing the Lemma. QED.

THEOREM 3. For $k \geq 2$ then $k(1 - 1/\log k)/\log k < d(k)$.

PROOF. We want to find a lower bound for $d(k)$ such that for all $n < d(k)$ we have

$$k > 2p_n - p_{n+1}. \qquad (2.16)$$

For $n \geq 20$ and using Lemma 5 we establish the following

$$2p_n - p_{n+1} < n(\log n + \log\log n + 1/2). \qquad (2.17)$$

From (2.17) we now replace (2.16) with a stronger inequality

$$k > n(\log n + \log\log n + 1/2). \qquad (2.18)$$

If $n = k(1 - 1/\log k)/\log k$, then by Lemma 6 (2.18) and hence (2.16) hold. Therefore $d(k) > n = k(1 - 1/\log k)/\log k$ establishing a lower bound for $d(k)$.
From Table 2 we see that Theorem 3 holds if $2 \le k \le 92$. QED.

THEOREM 4. For $19 \le k \le 286663$ then $d(k) < k/(\log k - \log\log k)$.

PROOF. To prove this Theorem we must divide this proof into four cases.
Case 1. $6036 < k \le 286663$
Case 2. $388 < k \le 6036$
Case 3. $193 < k \le 388$
Case 4. $19 < k \le 193$
In Cases 1-3 we will need to use the Pascal program called Verification which is found in Section 3, whereas for Case 4 we will use Table 1.
Case 1. We have

$$k < 2p_n - p_{n+1}. \qquad (2.19)$$

By using (1.3) we see that (2.19) will now become a more restrictive inequality

$$k < n(\log n + \log\log n - 1.5) + p_n - p_{n+1}. \qquad (2.20)$$

If we let $n = k/(\log k - \log\log k)$ and we use (1.6) then (2.20) becomes even more restrictive.

$$k < n(\log n + \log\log n - 1.5) - 652. \qquad (2.21)$$

By using the Pascal program, (2.21) is true for integer $k \in (6036, 286663]$ and hence (2.19) is true.
Case 2. Similar to Case 1 up to (2.20). However, we note that if $n = k/(\log k - \log\log k)$ then $n \in [93,922]$. Hence for n in this range we have $p_{n+1} - p_n \le 34$.
Hence in this range (2.20) becomes

$$k < n(\log n + \log\log n - 1.5) - 34. \qquad (2.22)$$

By using the Pascal program we can verify that (2.22) is true and hence (2.19) is true for $k \in (388, 6036]$.
Case 3. Similar to Case 2 except that $n \in [54, 92]$ and

$p_{n+1} - p_n \le 14$.

Case 4. We look at Table 1. QED.

LEMMA 7. Let $k \ge 286664$, $c = 2.531$ and $y = (\log(\log(k)-c)-c)/(\log(k)-c)$, then the maximum value of y is approximately .0292756256.

PROOF. $k = 286664$ then $y = -0.02241285$

$k \to \infty$ then $y \to 0$

Using elementary calculus we set $y' = 0$ an see that $k = \exp(e^{c+1}+c)$ For this k we note that $y'' < 0$ and hence we have a relative maximum; the value of y was determined by using a symbolic language called Maple. QED.

LEMMA 8. If $k \ge 286664$ and $c = 2.531$, let $n = k/(\log(k)-c)$ then $k < n(\log n + \log\log n - 2.500539232)$.

PROOF. Suppose not; then

$$k \ge n(\log n + \log\log n - 2.500539232). \tag{2.23}$$

After several manipulations of (2.23) we now have

$$0 \ge \log\log(k/(\log(k)-c))-\log(\log(k)-c)+c-2.500539232. \tag{2.24}$$

By grouping the terms containing log functions we see after several steps that (2.24) becomes

$$0 \ge \log(1+(c-\log(\log(k)-c))/(\log(k)-c))+c-2.500539232. \tag{2.25}$$

By Lemma 7 we see immediately that the smallest possible value for the second term within the outer log function is approximately .970724375. And hence (2.25) is false thereby establishing the Lemma. QED.

LEMMA 9. For $n \ge 28567$ then $2p_n - p_{n+1} > n(\log n + \log\log n - 2.500539232)$.

PROOF. Using (1.3) and (1.4) we have

$$2p_n - p_{n+1} > 2n(\log n + \log\log n - 3/2) - (n+1)(\log(n+1) - \log\log(n+1) - 1/2). \tag{2.26}$$

After several simple algebraic manipulations and using (1.7), (2.26) becomes

$$2p_n - p_{n+1} > n(\log n + \log\log n - 2.500003412) - n\log(1+1/n) - \log(n+1) - \log\log(n+1). \tag{2.27}$$

Using (1.8) we have the desired result. QED.

THEOREM 5. For $k \geq 286664$ then $d(k) < k/(\log k - 2.531)$.

PROOF. For $k \geq 286664$, we want to find an upper bound to $d(k)$ such that for all $n \geq d(k)$ we have

$$k < 2p_n - p_{n+1}. \tag{2.28}$$

For $n \geq 28567$ and Lemma 9 we have the following

$$n(\log n + \log\log n - 2.500539232) < 2p_n - p_{n+1}. \tag{2.29}$$

From (2.29) we replace (2.28) with a more restrictive inequality

$$k < n(\log n + \log\log n - 2.500539232). \tag{2.30}$$

Choose $n = k/(\log k - 2.531)$. Then by Lemma 8, (2.30) and hence (2.28) still hold. Therefore $d(k) < n = k/(\log k - 2.531)$, establishing an upper bound for $d(k)$. QED.

3. COMPUTER PROGRAMS AND TABLES.

The computer program called Verification was written in Turbo Pascal. Tables 1 and 2 were produced by other Pascal programs.

```
program Verification;
var  i:integer;
     j:longint;
     answer,k,n:real;
begin
     for  i:=194 to 388 do begin
          k:=i;
          n:=k/(ln(k)-ln(ln(k)));
          answer:=n*(ln(n)+ln(ln(n))-1.5)-14-k;
          if ( answer<0 ) then writeln(k,answer)
     end;
     for  i:=389 to 6036 do begin
          k:=i;
          n:=k/(ln(k)-ln(ln(k)));
          answer:=n*(ln(n)+ln(ln(n))-1.5)-34-k;
          if ( answer<0 ) then writeln(k,answer)
     end;
     for  j:=6037 to 286663 do begin
          k:=j;
          n:=k/(ln(k)-ln(ln(k)));
          answer:=n*(ln(n)+ln(ln(n))-1.5)-652-k;
          if ( answer<0 ) then writeln(k,answer)
     end
end.
```

TABLE 1

k	d(k)	f(k)	k	d(k)	f(k)	k	d(k)	f(k)	k	d(k)	f(k)
1	3		50	17	19.6235	99	31	32.2462	148	37	43.6793
2	3	1.8874	51	17	19.9007	100	31	32.4887	149	37	43.9046
3	5	2.9864	52	17	20.1768	101	31	32.7308	150	37	44.1297
4	5	3.7748	53	17	20.4519	102	31	32.9723	151	38	44.3545
5	5	4.4109	54	17	20.7258	103	31	33.2135	152	38	44.5790
6	5	4.9646	55	19	20.9987	104	31	33.4542	153	38	44.8032
7	5	5.4680	56	19	21.2706	105	31	33.6944	154	38	45.0272
8	5	5.9376	57	19	21.5415	106	31	33.9343	155	38	45.2510
9	7	6.3828	58	19	21.8115	107	31	34.1737	156	38	45.4745
10	7	6.8094	59	19	22.0805	108	31	34.4127	157	38	45.6977
11	7	7.2211	60	19	22.3486	109	31	34.6513	158	38	45.9207
12	7	7.6206	61	19	22.6157	110	31	34.8894	159	39	46.1434
13	7	8.0098	62	19	22.8820	111	31	35.1272	160	39	46.3659
14	7	8.3901	63	20	23.1475	112	31	35.3646	161	40	46.5881
15	9	8.7626	64	20	23.4120	113	31	35.6016	162	40	46.8101
16	9	9.1282	65	20	23.6758	114	31	35.8382	163	40	47.0319
17	10	9.4877	66	20	23.9387	115	31	36.0744	164	40	47.2534
18	10	9.8415	67	22	24.2009	116	31	36.3103	165	40	47.4746
19	10	10.1903	68	22	24.4623	117	31	36.5457	166	40	47.6957
20	10	10.5344	69	22	24.7229	118	31	36.7808	167	41	47.9165
21	10	10.8742	70	22	24.9828	119	31	37.0155	168	41	48.1370
22	10	11.2100	71	22	25.2419	120	31	37.2499	169	41	48.3574
23	10	11.5421	72	22	25.5003	121	31	37.4839	170	41	48.5775
24	10	11.8707	73	22	25.7580	122	31	37.7176	171	43	48.7974
25	12	12.1961	74	22	26.0151	123	32	37.9509	172	43	49.0170
26	12	12.5183	75	23	26.2714	124	32	38.1838	173	43	49.2365
27	12	12.8377	76	23	26.5271	125	33	38.4164	174	43	49.4557
28	12	13.1544	77	24	26.7821	126	33	38.6487	175	43	49.6747
29	12	13.4684	78	24	27.0365	127	33	38.8806	176	43	49.8934
30	12	13.7800	79	24	27.2902	128	33	39.1122	177	43	50.1120
31	12	14.0892	80	24	27.5433	129	35	39.3435	178	43	50.3303
32	12	14.3962	81	25	27.7958	130	35	39.5745	179	43	50.5485
33	13	14.7010	82	25	28.0477	131	35	39.8051	180	43	50.7664
34	13	15.0038	83	25	28.2990	132	35	40.0354	181	43	50.9841
35	13	15.3046	84	25	28.5497	133	35	40.2654	182	43	51.2016
36	13	15.6035	85	25	28.7999	134	35	40.4951	183	43	51.4189
37	13	15.9006	86	25	29.0495	135	35	40.7244	184	43	51.6359
38	13	16.1959	87	25	29.2985	136	35	40.9535	185	43	51.8528
39	15	16.4896	88	25	29.5470	137	35	41.1822	186	43	52.0695
40	15	16.7816	89	25	29.7949	138	35	41.4107	187	47	52.2860
41	16	17.0721	90	25	30.0423	139	35	41.6389	188	47	52.5022
42	16	17.3611	91	25	30.2892	140	35	41.8667	189	47	52.7183
43	16	17.6485	92	25	30.5356	141	35	42.0943	190	47	52.9342
44	16	17.9346	93	26	30.7814	142	35	42.3216	191	47	53.1499
45	16	18.2193	94	26	31.0268	143	35	42.5486	192	47	53.3653
46	16	18.5026	95	26	31.2716	144	35	42.7753	193	47	53.5806
47	17	18.7847	96	26	31.5160	145	37	43.0017	194	47	53.7957
48	17	19.0655	97	26	31.7599	146	37	43.2278	195	47	54.0106
49	17	19.3451	98	26	32.0033	147	37	43.4537			

where $f(k) = k/(\log k - \log \log k)$

TABLE 2

k	d(k)	g(k)	k	d(k)	g(k)	k	d(k)	g(k)
1	3		32	12	6.5691	63	20	11.5357
2	3	-1.2773	33	13	6.7387	64	20	11.6885
3	5	0.2451	34	13	6.9075	65	20	11.8410
4	5	0.8040	35	13	7.0754	66	20	11.9931
5	5	1.1764	36	13	7.2426	67	22	12.1449
6	5	1.4797	37	13	7.4090	68	22	12.2963
7	5	1.7486	38	13	7.5747	69	22	12.4474
8	5	1.9971	39	15	7.7396	70	22	12.5982
9	7	2.2319	40	15	7.9039	71	22	12.7487
10	7	2.4568	41	16	8.0675	72	22	12.8989
11	7	2.6743	42	16	8.2305	73	22	13.0488
12	7	2.8858	43	16	8.3929	74	22	13.1984
13	7	3.0923	44	16	8.5547	75	23	13.3478
14	7	3.2948	45	16	8.7159	76	23	13.4968
15	9	3.4936	46	16	8.8766	77	24	13.6455
16	9	3.6894	47	17	9.0367	78	24	13.7940
17	10	3.8824	48	17	9.1963	79	24	13.9422
18	10	4.0730	49	17	9.3554	80	24	14.0902
19	10	4.2613	50	17	9.5140	81	25	14.2379
20	10	4.4476	51	17	9.6721	82	25	14.3853
21	10	4.6320	52	17	9.8297	83	25	14.5325
22	10	4.8148	53	17	9.9869	84	25	14.6794
23	10	4.9959	54	17	10.1436	85	25	14.8261
24	10	5.1756	55	19	10.2999	86	25	14.9726
25	12	5.3538	56	19	10.4558	87	25	15.1188
26	12	5.5308	57	19	10.6112	88	25	15.2648
27	12	5.7065	58	19	10.7663	89	25	15.4105
28	12	5.8811	59	19	10.9209	90	25	15.5560
29	12	6.0546	60	19	11.0752	91	25	15.7013
30	12	6.2271	61	19	11.2291	92	25	15.8464
31	12	6.3986	62	19	11.3826	93	26	15.9913

where $g(k)=(k/(\log k))(1 - 1/\log k)$

4. COMMENT.

There is a discrepancy between the value of d(8) found in [4] and the value of d(8) found in this paper. This author believes that the value of d(8) is 5 and not 7.

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