Internat. J. Math. & Math. Sci.
VOL. 19 NO. 1 (1996) 103-110

FIXED POINT THEOREMS FOR NON-SELF MAPS IN d-COMPLETE TOPOLOGICAL SPACES

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(Received July 7, 1994 and in revised form August 8, 1994)

ABSTRACT. Fixed point theorems are given for non-self maps and pairs of non-self maps defined on d-complete topological spaces.

KEY WORDS AND PHRASES. d-complete topological spaces, fixed points, non-self maps, pairs of mappings.
1991 SUBJECT CLASSIFICATION CODE. 47H10, 54H25.

1. INTRODUCTION.

Let (X, t) be a topological space and $d: X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$. X is said to be d-complete if $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies that the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent in (X, t). Complete metric spaces and complete quasi-metric spaces are examples of d-complete topological spaces. The d-complete semi-metric spaces form an important class of examples of d-complete topological spaces.

Let X be an infinite set and t any T_1 non-discrete first countable topology for X. There exists a complete metric d for X such that $t \leq t_d$ and the metric topology t_d is non-discrete. Now (X, t, d) is d-complete since $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies that $\{x_n\}_{n=1}^{\infty}$ is Cauchy in t_d. Thus, $x_n \to x$, as $n \to \infty$, in t_d and therefore in the topology t. The construction of t_d is given by T. L. Hicks and W. R. Crisler in [1].

Recently, T. L. Hicks in [2] and T. L. Hicks and B. E. Rhoades in [3] and [4] proved several metric space fixed point theorems in d-complete topological spaces. We shall prove additional theorems in this setting.

Let $T: X \to X$ be a mapping. T is ω-continuous at x if $x_n \to x$ implies $Tx_n \to Tx$ as $n \to \infty$. A real-valued function $G: X \to [0, \infty)$ is lower semi-continuous if and only if $\{x_n\}_{n=0}^{\infty}$ is a sequence in X and $\lim_{n\to\infty} x_n = p$ implies $G(p) \leq \liminf_{n\to\infty} G(x_n)$.

2. RESULTS.

In [2], Hicks gave the following result.

THEOREM ([2], Theorem 2): Suppose X is a d-complete Hausdorff topological space, $T: X \to X$ is ω-continuous and satisfies $d(Tx, T^2x) \leq k(d(x, Tx))$ for all $x \in X$, where $k: [0, \infty) \to [0, \infty)$, $k(0) = 0$, and k is non-decreasing. Then T has a fixed point if and only if there exists x in X with $\sum_{n=1}^{\infty} k^n(d(x, Tx)) < \infty$. In this case, $x_n = T^n x \to p = Tp$. [k is not assumed to be continuous and $k^2(a) = k(k(a))$.]

The following conditions are examined. Let $T: C \to X$ with C a closed subset of the d-complete topological space X and $C \subset T(C)$. Let $k: [0, \infty) \to [0, \infty)$ be such that $k(0) = 0$, k is non-decreasing, and

$$k(d(Tx, Ty)) \geq d(x, y) \tag{2.1}$$

for all x, y ∈ C, or

$$d(Tx, Ty) \geq k(d(x, y)) \tag{2.2}$$

for all x, y ∈ C, or

$$d(x, y) \geq k(d(Tx, Ty)) \tag{2.3}$$

for all x, y ∈ C, or

$$k(d(x, y)) \geq d(Tx, Ty) \tag{2.4}$$

for all x, y ∈ C.

It will be shown that condition (2.1) leads to a fixed point, but that the other three conditions do not guarantee a fixed point.

THEOREM 1. Suppose X is a d-complete Hausdorff topological space, C is a closed subset of X, and $T: C \to X$ is an open mapping with $C \subset T(C)$ which satisfies $d(x, y) \leq k(d(Tx, Ty))$ for all $x, y \in C$ where $k: [0, \infty) \to [0, \infty)$, $k(0) = 0$, and k is non-decreasing. Then T has a fixed point if and only if there exists $x_0 \in C$ with $\sum_{n=1}^{\infty} k^n(d(Tx_0, x_0)) < \infty$.

PROOF. Notice that the condition $d(x, y) \leq k(d(Tx, Ty))$ forces T to be one-to-one. Hence T^{-1} exists. Also, T is open implies that T^{-1} is continuous, and thus ω-continuous.

If $p = Tp$ then $\sum_{n=1}^{\infty} k^n(d(Tp, p)) = 0 < \infty$.

Suppose there exists $x_0 \in C$ such that $\sum_{n=1}^{\infty} k^n(d(Tx_0, x_0)) < \infty$. We know that T^{-1} exists, so let T_1 be T^{-1} restricted to C. Then $T_1 : C \to C$ and $d(T_1x, T_1y) \leq k(d(x, y))$ for all x, $y \in C$. Let $y = T_1x$. Then $d(T_1x, T_1^2x) \leq k(d(x, T_1x))$ for all $x \in C$. In particular,
$d(T_1x_0, T_1^2x_0) \leq k(d(x_0, T_1x_0)) \leq k^2(d(Tx_0, x_0))$. By induction,
$d(T_1^{n-1}x_0, T_1^nx_0) \leq k^n(d(Tx_0, x_0))$. Thus,

$$\sum_{k=1}^{\infty} d(T_1^{n-1}x_0, T_1^nx_0) \leq \sum_{k=1}^{\infty} k^n(d(Tx_0, x_0)) < \infty.$$

Since X is d-complete, $T_1^nx_0$ converges, say to p. Note that p is in C since C is closed. Now $T_1(T_1^nx_0) \to T_1p$ as $n \to \infty$ since T_1 is ω-continuous. But $T_1^{n+1}x_0 \to p$ as $n \to \infty$, and since limits are unique in X, $T_1p = p$. Now $T(T_1p) = T(p)$ and $T(T_1p) = p$ so $Tp = p$ and T has a fixed point.

COROLLARY 1. Suppose $T: C \to X$ where C is a closed subset of a d-complete Hausdorff symmetrizable topological space with $C \subset T(C)$. Suppose $d(x, y) \leq [d(Tx, Ty)]^p$ where $p > 1$ for all $x, y \in C$. If there exists $x_0 \in C$ such that $d(Tx_0, x_0) < 1$, then T has a fixed point.

PROOF. If $x \neq y$, $0 < d(x, y) \leq [d(Tx, Ty)]^p$ and $Tx \neq Ty$. Thus T is one-to-one and T^{-1} exists. Now $d(T^{-1}x, T^{-1}y) \leq [d(x, y)]^p$ implies that T^{-1} is continuous. Hence T must be open. Let x_0 be a point in C such that $d(Tx_0, x_0) < 1$. If $d(Tx_0, x_0) = 0$, then x_0 is a fixed point of T. Suppose $0 < d(Tx_0, x_0) < 1$. Let $k(t) = t^p$, and $t = d(Tx_0, x_0)$. Note that $(\alpha t)^p < \alpha t^p$ if $0 < \alpha < 1$. Since $t^p < t$, there is an $\alpha_1 \in (0, 1)$ such that $t^p = \alpha_1 t$. Now $(t^p)^p < t^p$ and there is an $\alpha_2 \in (0, 1)$ such that $t^{2p} = \alpha_2 t^p$. But $\alpha_2 t^p = t^{2p} = (t^p)^p = (\alpha_1 t)^p < \alpha_1 t^p$. Hence $\alpha_2 < \alpha_1$. Now $t^{2p} = \alpha_2 t^p = \alpha_2 \alpha_1 t < (\alpha_1)^2 t$. Assume $t^{np} < \alpha_1^n t$. Then $t^{(n+1)p} = (t^{np})^p < (\alpha_1^n t)^p = \alpha_1^{np} t^p = \alpha_1^{np} \alpha_1 t = \alpha_1^{(n+1)p} t$. Hence, by induction, $t^{np} < \alpha_1^n t$ for all natural numbers n. Therefore,

$$\sum_{n=1}^{\infty} k^n(d(Tx_0, x_0)) = \sum_{n=1}^{\infty} [d(Tx_0, x_0)]^{np} = \sum_{n=1}^{\infty} t^{np} < \sum_{n=1}^{\infty} \alpha_1^n t < \infty$$

since $0 < \alpha_1 < 1$. Applying Theorem 1, we get that T has a fixed point.

If T is not open one could check the following condition.

THEOREM 2. Let X be a d-complete Hausdorff topological space, C be a closed subset of X, $T : C \to X$ with $C \subset T(C)$. Suppose there exists $k: [0, \infty) \to [0, \infty)$ such that $k(d(Tx, Ty)) \geq d(x, y)$ for all $x, y \in C$, k is non-decreasing, $k(0) = 0$, and there exists $x_0 \in C$ such that $\sum_{n=1}^{\infty} k^n(d(Tx_0, x_0)) < \infty$. If $G(x) = d(Tx, x)$ is lower semi-continuous on C then T has a fixed point.

PROOF. If $x \neq y$, $0 < d(x, y) \leq k(d(Tx, Ty))$ so that $d(Tx, Ty) \neq 0$. Hence T is one-to-one and T^{-1} exists. Let T_1 be T^{-1} restricted to C. Now $T_1 : C \to C$ and for $x \in C$, $d(x, T_1x) \leq k(d(Tx, x))$, $d(T_1x, T_1^2x,) \leq k(d(x, T_1x)) \leq k^2(d(Tx, x))$. By induction, $d(T_1^{n-1}x, T_1^n x) \leq k^n(d(Tx, x))$. There exists $x_0 \in C$ with $\sum_{n=1}^{\infty} k^n(d(Tx_0, x_0)) < \infty$ implies $\sum_{n=1}^{\infty} d(T_1^{n-1}x_0, T_1^n x_0) < \infty$. Since X is d-complete there exists $p \in X$ such that $T_1^n x_0 \to p$ as $n \to \infty$. Note that $p \in C$ since $T_1^n x_0 \in C$ for all n and C is closed. Now $G(x) = d(Tx, x)$ is lower semi-continuous on C gives $G(p) \leq \liminf G(T_1^n x_0)$ or $d(Tp, p) \leq \liminf d(T_1^{n-1}x_0, T_1^n x_0,) = 0$. Thus $Tp = p$.

In [5], Hicks gives several examples of functions k which satisfy the condition of theorem 1 of that paper. These examples, with a slight modification, carry over to the non-self map case. The non-self map version of Example 1 is given for completeness. The other examples carry over in a similar manner.

EXAMPLE 1. Suppose $0 < \lambda < 1$. Let $k(t) = \lambda t$ for $t \geq 0$. If $d(x, y) \leq \lambda d(Tx, Ty)$, T is open since T^{-1} exists and is continuous. Let $x \in C$. There exists $y \in C$ such that $Ty = x$. Now $d(x, y) = d(Ty, y) \leq \lambda d(T^2y, Ty)$ and $\sum_{n=1}^{\infty} k^n(d(Ty, y)) \leq \sum_{n=1}^{\infty} \lambda^n d(T^2y, Ty) < \infty$. Applying Theorem 1 we get a fixed point for T. (Note: $d(x, y) \leq \lambda d(Tx, Ty)$ for $0 < \lambda < 1$ is equivalent to $d(Tx, Ty) \geq \alpha d(x, y)$ for $\alpha > 1$.)

The following examples show that conditions (2.2), (2.3) and (2.4) do not guarantee fixed points.

EXAMPLE 2. Let $\mathbb{R}$ denote the real numbers and $CB(\mathbb{R}, \mathbb{R})$ denote the collection of all bounded and continuous functions which map $\mathbb{R}$ into $\mathbb{R}$. Let

$$C = \{f \in CB(\mathbb{R}, \mathbb{R}) : f(t) = 0 \text{ for all } t \leq 0 \text{ and } \lim_{t\to\infty} f(t) \geq 1\}.$$

Define $T : C \to CB(\mathbb{R}, \mathbb{R})$ by $Tf(t) = \frac{1}{2}f(t+1)$ and let $k(t) = \frac{t}{3}$. Then $d(Tf, Tg) = \frac{1}{2}d(f, g) \geq k(d(f, g))$. k satisfies condition (2.2) but, as shown in [6], T does not have a fixed point.

EXAMPLE 3. Let $T : [1, \infty) \to [0, \infty)$ be defined by $Tx = x - \frac{1}{x}$ and let $k(t) = \frac{t}{2}$. Then $d(Tx, Ty) \leq 2\, d(x, y)$ or $d(x, y) \geq k(d(Tx, Ty))$. k satisfies condition (2.3) but T does not have a fixed point.

EXAMPLE 4. Let c_0 denote the collection of all sequences that converge to zero. Let $C = \{x \in c_0 : \|x\| = 1 \text{ and } x_0 = 1\}$. Define $T : C \to c_0$ by $Tx = y$ where $y_n = x_{n+1}$, $n = 0, 1, 2, \ldots$, and let $k(t) = 2t$. Then $d(Tx, Ty) = d(x, y) \leq 2\, d(x, y) = k(d(x, y))$ for all $x, y \in C$. k satisfies condition (2.4) but, as shown in [6], T does not have a fixed point.

The following theorems were motivated by the work of Hicks and Rhoades [3].

THEOREM 3. Let C be a compact subset of a Hausdorff topological space (X, t) and $d : X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$. Suppose $T : C \to X$ with $C \subset T(C)$, T and $G(x) = d(x, Tx)$ are both continuous, and $d(Tx, T^2x) > d(x, Tx)$ for all $x \in T^{-1}(C)$ with $x \neq Tx$. Then T has a fixed point in C.

PROOF. C is a compact subset of a Hausdorff space so it is closed. T is continuous so $T^{-1}(C)$ is closed and hence is compact since $T^{-1}(C) \subset C$. $G(x)$ is continuous so it attains its minimum on $T^{-1}(C)$, say at z. Now $z \in C \subset T(C)$ so there exists $y \in T^{-1}(C)$ such that $Ty = z$. If $y \neq z$ then $d(z, Tz) = d(Ty, T^2y) > d(y, Ty)$, a contradiction. Thus $y = z = Ty$ is a fixed point of T.

THEOREM 4. Let C be a compact subset of a Hausdorff topological space (X, t) and $d : X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$. Suppose $T : C \to X$ with $C \subset T(C)$, T and $G(x) = d(x, Tx)$ are both continuous, $f : [0, \infty) \to [0, \infty)$ is continuous and $f(t) > 0$ for $t \neq 0$. If we know that $d(Tx, T^2x) \leq \lambda\, f(d(x, Tx))$ for all $x \in T^{-1}(C)$ implies T has a fixed point where $0 < \lambda < 1$, then $d(Tx, T^2x) < f(d(x, Tx))$ for all $x \in T^{-1}(C)$ such that $f(d(x, Tx)) \neq 0$ gives a fixed point.

PROOF. C is a compact subset of a Hausdorff space so it is closed. T is continuous gives that $T^{-1}(C)$ is closed, and $T^{-1}(C) \subset C$ so $T^{-1}(C)$ is compact. Suppose $x \neq Tx$ for all $x \in T^{-1}(C)$. Then $d(x, Tx) > 0$ so that $f(d(x, Tx)) > 0$ for all $x \in T^{-1}(C)$. Define $P(x)$ on $T^{-1}(C)$ by $P(x) = \frac{d(Tx\ T^2x)}{f(d(x, Tx))}$. P is continuous since T, f and $G(x)$ are continuous. Therefore P attains its maximum on $T^{-1}(C)$, say at z. $P(x) \leq P(z) < 1$ so $d(Tx, T^2x) \leq P(z)f(d(x, Tx))$ and T must have a fixed point.

THEOREM 5. Let C be a compact subset of a Hausdorff topological space (X, t) and $d : X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$. Suppose $T : C \to X$ with $C \subset T(C)$, T and $G(x) = d(x, Tx)$ are both continuous, $f : [0, \infty) \to [0, \infty)$ is continuous and $f(t) > 0$ for $t \neq 0$. If we know that $d(Tx, T^2x) \geq \lambda\, f(d(x, Tx))$ for all $x \in T^{-1}(C)$ implies T has a fixed point where $\lambda > 1$, then $d(Tx, T^2x) > f(d(x, Tx))$ for all $x \in T^{-1}(C)$ such that $f(d(x, Tx)) \neq 0$ gives a fixed point.

PROOF. C is a compact subset of a Hausdorff space so it is closed. T is continuous gives that $T^{-1}(C)$ is closed and hence compact, since $T^{-1}(C) \subset C$. Suppose $x \neq Tx$ for all $x \in T^{-1}(C)$. Then $d(x, Tx) > 0$ and $f(d(x, Tx)) > 0$. Define $P(x) = \frac{d(Tx, T^2x)}{f(d(x\ Tx))}$. P is continuous since T, f and G are continuous. P attains it minimum on $T^{-1}(C)$, say at z. $P(x) \geq P(z) > 1$ so $d(Tx, T^2x) \geq P(z)f(d(x, Tx))$ and T must have a fixed point.

Theorems 6, 7 and 8 are generalizations of theorems by Kang [7]. The following family of real functions was originally introduced by M. A. Khan, M. S. Khan, and S. Sessa in [8]. Let Φ denote the family of all real functions $\phi : (\mathbb{R}^+)^3 \to \mathbb{R}^+$ satisfying the following conditions:

(C_1) ϕ is lower-semicontinuous in each coordinate variable,

(C_2) Let $v, w \in \mathbb{R}^+$ be such that either $v \geq \phi(v, w, w)$ or $v \geq \phi(w, v, w)$. Then $v \geq hw$, where $\phi(1, 1, 1) = h > 1$.

THEOREM 6. Let (X, t, d) be a d-complete topological space where d is a continuous symmetric. Let A and B map C, a closed subset of X, into (onto) X such that $C \subset A(C)$, $C \subset B(C)$, and $d(Ax, By) \geq \phi(d(Ax, x), d(By, y), d(x, y))$ for all x, y in C where $\phi \in \Phi$. Then A and B have a common fixed point in C.

PROOF. Fix $x_0 \in C$. Since $C \subset A(C)$ there exists $x_1 \in C$ such that $Ax_1 = x_0$. Now $C \subset B(C)$ so there exists $x_2 \in C$ such that $Bx_2 = x_1$. Build the sequence $\{x_n\}_{n=0}^{\infty}$ by $Ax_{2n+1} = x_{2n}$, $Bx_{2n+2} = x_{2n+1}$. Now if $x_{2n+1} = x_{2n}$ for some n, the x_{2n+1} is a fixed point of A. Then

$$\begin{aligned} d(x_{2n+1}, x_{2n+1}) &= d(x_{2n}, x_{2n+1}) \\ &= d(Ax_{2n+1}, Bx_{2n+2}) \\ &\geq \phi(d(Ax_{2n+1}, x_{2n+1}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})) \\ &= \phi(0, d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})) \end{aligned}$$

By property (C_2), $d(x_{2n}, x_{2n+1}) \geq h\,d(x_{2n+1}, x_{2n+2})$. Hence, $x_{2n+1} = x_{2n+2}$ and $Bx_{2n+1} = Bx_{2n+2} = x_{2n+1}$. Therefore x_{2n+1} is a common fixed point of A and B. Now if $x_{2n+1} = x_{2n+2}$ for some n, then $Bx_{2n+2} = Bx_{2n+1} = x_{2n+2}$. Then

$$\begin{aligned} d(x_{2n+2}, x_{2n+1}) &= d(Ax_{2n+3}, Bx_{2n+2}) \\ &\geq \phi(d(Ax_{2n+3}, x_{2n+3}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+3}, x_{2n+2})) \\ &= \phi(d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+3}, x_{2n+2})). \end{aligned}$$

By property (C_2) , $d(x_{2n+1}, x_{2n+2}) \geq h\,d(x_{2n+2}, x_{2n+3})$ or $x_{2n+2} = x_{2n+3}$. Thus $Ax_{2n+2} = Ax_{2n+3} = x_{2n+2}$ and x_{2n+2} is a fixed point of A also.

Suppose $x_n \neq x_{n+1}$ for all n. Then

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(Ax_{2n+1}, Bx_{2n+2}) \\ &\geq \phi(d(Ax_{2n+1}, x_{2n+1}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})) \\ &= \phi(d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})). \end{aligned}$$

Again by (C_2), $d(x_{2n}, x_{2n+1}) \geq h\, d(x_{2n+1}, x_{2n+2})$ or $d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{h} d(x_{2n}, x_{2n+1})$. Also,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Ax_{2n+3}, Bx_{2n+2}) \\ &\geq \phi(d(Ax_{2n+3}, x_{2n+3}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+3}, x_{2n+2})) \\ &= \phi(d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+3}, x_{2n+2})). \end{aligned}$$

By (C_2) we get $d(x_{2n+2}, x_{2n+3}) \leq \frac{1}{h} d(x_{2n+1}, x_{2n+2})$. Induction gives $d(x_{n+1}, x_{n+2}) \leq \left(\frac{1}{h}\right)^{n+1} d(x_0, x_1)$. Thus $\sum_{n=1}^{\infty} d(x_{n+1}, x_{n+2}) \leq \sum_{n=1}^{\infty} \left(\frac{1}{h}\right)^{n+1} d(x_0, x_1) < \infty$. X is d-complete so $x_n \to p$ as $n \to \infty$ where $p \in C$, since C is closed. We also have $x_{2n} \to p$ and $x_{2n+1} \to p$ as $n \to \infty$. This gives $Ax_{2n+1} \to p$ and $Bx_{2n+2} \to p$ as $n \to \infty$. Since $p \in C$, $p \in A(C)$ and $p \in B(C)$, so there exist $v, w \in C$ such that $Av = p$ and $Bw = p$. Now

$$\begin{aligned} d(x_{2n}, p) &= d(Ax_{2n+1}, Bw) \\ &\geq \phi(d(Ax_{2n+1}, x_{2n+1}), d(Bw, w), d(x_{2n+1}, w)). \end{aligned}$$

Since ϕ is lower-semicontinuous, letting $n \to \infty$ gives $d(p, p) \geq \phi(0, d(p, w), d(p, w))$ and by (C_2) we have $0 \geq h\, d(p, w)$. Hence $p = w$. Also,
$d(p, x_{2n+1}) = d(Av, Bx_{2n+1}) \geq \phi(d(Av, v), d(Bx_{2n+2}, x_{2n+2}), d(v, x_{2n+1}))$.
Letting $n \to \infty$ gives $d(p, p) \geq \phi(d(p, v), 0, d(v, p))$ or, by (C_2), $0 \geq h\, d(p, v)$. Hence $p = v$. Therefore, $Ap = Av = p = Bw = Bp$.

COROLLARY 2. Let A and B map C, a closed subset of X, into (onto) X such that $C \subset A(C)$, $C \subset B(C)$, and $d(Ax, By) \geq a\, d(Ax, x) + b\, d(By, y) + c\, d(x, y)$ for all $x, y \in C$, where a, b, and c are non-negative real numbers with $a < 1$, $b < 1$, and $a + b + c > 1$. Then A and B have a common fixed point in C.

The proof of Corollary 2 is identical to the proof of Corollary 2.3 in [9].

In [7], Kang defined Φ^* to be the family of all real functions $\varphi \to (\mathbb{R}^+)^3 \to \mathbb{R}^+$ satisfying condition (C_1) and the following condition:
(C_3) Let $v, w \in \mathbb{R}^+ - \{0\}$ be such that either $v \geq \varphi(v, w, w)$ or $v \geq \varphi(w, v, w)$. Then $v \geq hw$, where $\varphi(1, 1, 1) = h > 1$. Kang showed that the family Φ^* is strictly larger than the family Φ.

THEOREM 7. Let (X, t, d) be a d-complete Hausdorff topological space where d is a continuous symmetric. If A and B are continuous mappings from C, a closed subset of X, into X such that $C \subset A(C)$, $C \subset B(C)$, and $d(Ax, By) \geq \varphi(d(Ax, x), d(By, y), d(x, y))$ for all $x, y \in C$ such that $x \neq y$ where $\varphi \in \Phi^*$, then A or B has a fixed point or A and B have a common fixed point.

PROOF. Let $\{x_n\}_{n=0}^{\infty}$ be defined as in the proof of Theorem 6. If $x_n = x_{n+1}$ for some n then A or B has a fixed point. Suppose $x_n \neq x_{n+1}$ for all n. As in the proof of Theorem 6, $x_n \to p$ as $n \to \infty$. Now $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=0}^{\infty}$ are subsequences of $\{x_n\}_{n=1}^{\infty}$ and hence each converges to p. Since A and B are continuous, $Ax_{2n+1} = x_{2n} \to Ap$ and $Bx_{2n+2} = x_{2n+1} \to Bp$. Limits in X are unique, because X is Hausdorff, so $Ap = p = Bp$.

COROLLARY 3. Let A and B be continuous mappings from C, a closed subset of X, into X satisfying $C \subset A(C)$, $C \subset B(C)$ and $d(Ax, By) \geq h \min\{d(Ax, x), d(By, y), d(x, y)\}$ for all x, $y \in C$ with $x \neq y$ where $h > 1$. Then A or B has a fixed point or A and B have a common fixed point.

PROOF. Note that $\varphi(t_1, t_2, t_3) = h \min\{t_1, t_2, t_3\}$, $h > 1$ is in Φ^*. Apply Theorem 7.

If A = B in Corollary 3 we get a generalization of Theorem 3 in [9].

Boyd and Wong [10] call the collection of all real functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the following conditions Ψ:

(C_4) ψ is upper-semicontinuous and non-decreasing,

(C_5) $\psi(t) < t$ for each $t > 0$.

THEOREM 8. Let (X, t, d) be a d-complete symmetric Hausdorff topological space. If A and B are continuous mappings from C, a closed subset of X, into X such that $C \subset A(C)$, $C \subset B(C)$, and $\psi(d(Ax, By)) \geq \min\{d(Ax, x), d(By, y), d(x, y)\}$ for all $x, y \in C$ where $\psi \in \Psi$ and $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, then either A or B has a fixed point or A and B have a common fixed point.

PROOF. Let $\{x_n\}_{n=0}^{\infty}$ be defined as in the proof of Theorem 6. If $x_n = x_{n+1}$ for some n then A or B has a fixed point. Suppose $x_n \neq x_{n+1}$ for all n. Then

$$\begin{aligned} \psi(d(x_{2n}, x_{2n+1})) &= \psi(d(Ax_{2n+1}, Bx_{2n+2})) \\ &\geq \min\{d(Ax_{2n+1}, x_{2n+1}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})\} \\ &= \min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})\} \\ &= d(x_{2n+1}, x_{2n+2}) \end{aligned}$$

since $\psi(t) < t$ for all $t > 0$.

Similarly, $d(x_{2n+2}, x_{2n+3}) \leq \psi(d(x_{2n+1}, x_{2n+2}))$ and hence $d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1}))$ for each n. Since ψ is non-decreasing, $d(x_{n+1}, x_{n+2}) \leq \psi^n(d(x_0, x_1))$. Now

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=0}^{\infty} \psi^n(d(x_0, x_1)) < \infty.$$

The space X is d-complete so there exists $p \in C$ such that $x_n \to p$ as $n \to \infty$. The mappings A and B are continuous so $Ax_{2n+1} = x_{2n} \to Ap$ and $Bx_{2n+2} = x_{2n+1} \to Bp$. Limits are unique so $Ap = p = Bp$.

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