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COMMON FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS

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ABSTRACT. By using a compatibility condition due to Jungck we establish some common fixed point theorems for four mappings on complete and compact metric spaces These results also generalize a theorem of Sharma and Sahu

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1. INTRODUCTION.

The following theorem was given by Sharma and Sahu [3]

THEOREM 1.1. Let (X,d) be a complete metric space and let S, T and A be three continuous mappings of X into itself satisfying the conditions

(i) $SA = AS, \qquad TA = AT,$

(ii) $S(X) \subseteq A(X), \qquad T(X) \subseteq A(X),$

and the inequality

$$\begin{aligned}[d(Sx,Ty)]^2 \leq\ & a_1 d(Ax,Sx)d(Ay,Ty) + a_2 d(Ax,Ty)d(Ay,Sx) \\ & + a_3 d(Ax,Sx)d(Ax,Ty) + a_4 d(Sx,Ay)d(Ty,Ay) \\ & + a_5 [d(Ax,Ay)]^2\end{aligned}$$

for all x,y in X, where $a_i \geq 0$ for $i = 1,2,3,4,5$, $a_1 + a_4 + a_5 < 1$ and $2a_1 + 3a_3 + 2a_5 < 2$, then S,T and A have a unique common fixed point in X

Unfortunately, the theorem as stated is incorrect First of all the only restriction on a_2 is that $a_2 \geq 0$ This is clearly false In fact the inequality $a_1 + a_4 + a_5 < 1$ should have been $a_2 + a_4 + a_5 < 1$ This was only a typing error

More seriously, on examination of the proof it is seen that the inequality $2a_1 + 3a_3 + 2a_5 < 2$ was obtained by reading $[d(Ax_{2n}, Ax_{2n+2})]^2$ as $[d(Ax_{2n+1}, Ax_{2n}]^2$ This inequality should have been $a_1 + 2a_3 + a_5 < 1$ The proof continues with a 'similarity ', but on following through it is seen that this time the inequality $a_1 + 2a_4 + a_5 < 1$ is required

2. PRELIMINARIES.

The following definitions were given by Sessa [2] and Jungck [1] respectively

DEFINITION 2.1. [2] Self-mappings f and g on a metric space (X, d) are said to weakly commute if and only if $d(fgx, gfx) \le d(fx, gx)$ for all x in X

DEFINITION 2.2. [1] Self-mappings f and g on a metric space (X, d) are said to be compatible if and only if whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X, then $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$

Note that f and g commute, then it is easily seen that they weakly commute and if f and g weakly commute, then they are compatible However, weakly commuting mappings do not necessarily commute and compatible mappings do not necessarily weakly commute For examples, see [1] and [2]

PROPOSITION 2.1. [1] Let f and g be compatible self-mappings on a metric space (X, d) with the properties

(i) if $f(t) = g(t)$, then $fg(t) = gf(t)$,

(ii) $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$

Then $\lim_{n\to\infty} gf(x_n) = f(t)$ if f is continuous

We now prove the following theorem

THEOREM 2.1. Let f and g be self-mappings of the set $X = \{x, x'\}$ with any metric d If the range of g contains the range of f, then the following statements are equivalent

(i) f and g commute,

(ii) f and g weakly commute,

(iii) f and g are compatible

PROOF. The implication that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is clear It is therefore sufficient to show that (iii) $\Rightarrow$ (i) The four self-mappings on X are as follows

$$Ax = x\,, \quad Ax' = x'\,, \quad Bx = Bx' = x\,, \quad Cx = Cx' = x'\,, \quad Dx = x'\,, \quad Dx' = x\,.$$

We note that A is the identity mapping on X and since we only have to prove that compatible mappings commute, we may suppose that $f \ne g$, $f \ne A$ and $g \ne A$ Further, since $f(X) \subseteq g(X)$, we only have to consider the following two cases

$$\text{(a) } B = f\,, \quad D = g \qquad \text{and} \qquad \text{(b) } C = f\,, \quad D = g\,.$$

If f and g are compatible and (a) holds, then $Bx' = Dx'$ implies that $BDx' = DBx'$ by Proposition 2 1 Consequently, $x = x'$, a contradiction

Similarly, if (b) holds, then $Dx = Cx$ implies that $DCx = CDx$ by Proposition 2.1 and again $x = x'$, a contradiction The conditions of the theorem can therefore hold only if either $f = g$ or $f = A$ or $g = A$ and the commutativity of f and g follows This completes the proof

3. MAIN RESULTS.

Let A, B, S and T be self-mappings of a metric space (X, d) such that

$$A(X) \subseteq T(X) \qquad \text{and} \qquad B(X) \subseteq S(X)\,, \tag{3 1}$$

$$\begin{aligned}[d(Ax, By)]^2 \le\ & c_1 \max\{[d(Sx, Ax)]^2, [d(Ty, By)]^2, [d(Sx, Ty)]^2\} \\ & + c_2 \max\{d(Sx, Ax)d(Sx, By), d(Ax, Ty)d(By, Ty)\} \\ & + c_3 d(Sx, By)d(Ty, Ax)\end{aligned} \tag{3 2}$$

for all x, y in X, where $c_1, c_2, c_3 \geq 0$, $c_1 + 2c_2 < 1$ and $c_1 + c_3 < 1$

Now let x_0 be an arbitrary point in X Then since (3 1) holds, we can define a sequence

$$\{y_n\} = \{Ax_0, Bx_1, Ax_2, Bx_3, \ldots, Ax_{2n}, Bx_{2n+1}, \ldots\} \tag{3 3}$$

inductively such that $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$, $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n = 1, 2, \ldots$

For simplicity, let $d_n = d(y_n, y_{n-1})$ for $n = 0, 1, 2, \ldots$ We need the following lemma for our main theorem

LEMMA 3.1. The sequence $\{y_n\}$ defined in (3 3) is a Cauchy sequence

PROOF. Using inequality (3 2) we have

$$\begin{aligned} d_{2n+1}^2 &= [d(y_{2n+1}, y_{2n+2})]^2 = [d(Ax_{2n}, Bx_{2n+1})]^2 \\ &\leq c_1 \max\{[d(Sx_{2n}, Ax_{2n})]^2, [d(Tx_{2n+1}, Bx_{2n+1})]^2, [d(Sx_{2n}, Tx_{2n+1})]^2\} \\ &\quad + c_2 \max\{d(Sx_{2n}, Ax_{2n}) d(Sx_{2n}, Bx_{2n+1}), d(Ax_{2n}, Tx_{2n+1}) d(Bx_{2n+1}, Tx_{2n+1})\} \\ &\quad + c_3 d(Sx_{2n}, Bx_{2n+1}) d(Tx_{2n+1}, Ax_{2n}) \\ &\leq c_1 \max\{d_{2n}^2, d_{2n+1}^2\} + c_2 d_{2n} d(y_{2n}, y_{2n+2}) \\ &\leq c_1 \max\{d_{2n}^2, d_{2n+1}^2\} + c_2 [d_{2n}^2 + d_{2n} d_{2n+1}] \\ &\leq c_1 \max\{d_{2n}^2, d_{2n+1}^2\} + c_2 \left[\frac{3}{2} d_{2n}^2 + \frac{1}{2} d_{2n+1}^2\right] . \end{aligned} \tag{3 4}$$

If $d_{2n+1} > d_{2n}$, inequality (3 4) implies

$$d_{2n+1}^2 \leq \frac{3c_2}{2 - 2c_1 - c_2} d_{2n}^2 ,$$

a contradiction since

$$\frac{3c_2}{2 - 2c_1 - c_2} < 1 .$$

Thus $d_{2n+1} \leq d_{2n}$ and inequality (3.4) implies that

$$d_{2n+1} = d(y_{2n+1}, y_{2n+2}) \leq h d(y_{2n}, y_{2n+1}) = d_{2n} ,$$

where

$$h^2 = \frac{2c_1 + 3c_2}{2 - c_2} < 1 .$$

Similarly,

$$d_{2n}^2 = [d(y_{2n}, y_{2n+1})]^2 = [d(Ax_{2n}, Bx_{2n-1})]^2 \leq c_1 \max\{d_{2n-1}^2, d_{2n}^2\} + c_2 \left(\frac{3}{2} d_{2n-1}^2 + \frac{1}{2} d_{2n}^2\right)$$

and it follows as above that

$$d_{2n} = d(y_{2n}, y_{2n+1}) \leq h d(y_{2n-1}, y_{2n}) = d_{2n-1} .$$

Consequently,

$$d(y_{n+1}, y_n) \leq h d(y_n, y_{n-1}) ,$$

for $n = 1, 2, \ldots$ Since $h < 1$, this last inequality implies that $\{y_n\}$ is a Cauchy sequence in X

Now by replacing commuting mappings with compatible mappings, and by using four mappings as opposed to three, we prove the following generalization of the amended Theorem 1 1 which shows that only one of the mappings in Theorem 1 1 needs to be continuous

THEOREM 3.1. Let A, B, S and T be self-mappings on a complete metric space (X, d) satisfying conditions (3 1) and (3 2) and suppose that one of the mappings is continuous If A and B are compatible with S and T respectively, then A, B, S and T have a common fixed point z Further, z is the unique common fixed point of A and S and of B and T

PROOF. Since X is complete, it follows from Lemma 3 1 that the sequence (3 3) converges to a point z in X. The subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ must therefore also converge to z

Let us first of all suppose that S is continuous Since A and S are compatible, Proposition 2 1 implies that $\{S^2x_{2n}\}$ and $\{ASx_{2n}\}$ converge to Sz Using inequality (3 2) we have

$$\begin{aligned}[d(ASx_{2n}, Bx_{2n-1})]^2 \leq\ & c_1\max\{[d(S^2x_{2n}, ASx_{2n})]^2, [d(Tx_{2n-1}, Bx_{2n-1})]^2, [d(S^2x_{2n}, Tx_{2n-1})]^2\} \\ & + c_2\max\{d(S^2x_{2n}, ASx_{2n})d(S^2x_{2n}, Bx_{2n-1}), d(ASx_{2n}, Tx_{2n-1})d(Bx_{2n-1}, Tx_{2n-1})\} \\ & + c_3 d(S^2x_{2n}, Bx_{2n-1})d(Tx_{2n-1}, ASx_{2n}) \ .\end{aligned}$$

Letting n tend to infinity we have

$$[d(Sz, z)]^2 \leq (c_1 + c_3)[d(Sz, z)]^2$$

and since $c_1 + c_3 < 1$, it follows that $Sz = z$

Using inequality (3 2) again we have

$$\begin{aligned}[d(Az, Bx_{2n-1})]^2 \leq\ & c_1\max\{[d(Sz, Az)]^2, [d(Tx_{2n-1}, Bx_{2n-1})]^2, [d(Sz, Tx_{2n-1})]^2\} \\ & + c_2\max\{d(Sz, Az)d(Sz, Bx_{2n-1}), d(Az, Tx_{2n-1})d(Bx_{2n-1}, Tx_{2n-1})\} \\ & + c_3 d(Sz, Bx_{2n-1})d(Tx_{2n-1}, Az) \ .\end{aligned}$$

Letting n tend to infinity we have

$$[d(Az, z)]^2 \leq c_1[d(Sz, Az)]^2 = c_1[d(z, Az)]^2$$

and since $c < 1$, it follows that $Az = z$.

Now since $A(X) \subseteq T(X)$, there exists a point u in X such that $z = Az = Sz = Tu$. Then on applying inequality (3 2) to $[d(Az, Bu)]^2$, it follows that

$$[d(Az, Bu)]^2 = [d(z, Bu)]^2 \leq c_1[d(z, Bu)]^2 \ ,$$

which implies that $z = Bu$. Since B and T are compatible and $Tu = Bu = z$, we have from Proposition (2.1) that $TBu = BTu$ which implies that $Tz = Bz$. Again, on applying inequality (3.2) to $[d(Az, Bz)]^2$, it follows that

$$[d(Az, Bz)]^2 = [d(z, Bz)]^2 \leq (c_1 + c_3)[d(z, Bz)]^2$$

and so $z = Bz = Tz$. We have therefore proved that z is a common fixed point of A, B, S and T if S is continuous.

The same result of course holds if we suppose that T is continuous instead of S.

Now suppose that A is continuous. Since A and S are compatible, it follows from Proposition (2.1) that the sequences $\{A^2x_{2n}\}$ and $\{SAx_{2n}\}$ converge to Az From inequality (3 2) we have

$$\begin{aligned}[d(A^2x_{2n}, Bx_{2n-1})]^2 \leq\ & c_1\max\{[d(SAx_{2n}, A^2x_{2n})]^2, [d(Tx_{2n-1}, Bx_{2n-1})]^2, [d(SAx_{2n}, Tx_{2n-1})]^2\} \\ & + c_2\max\{d(SAx_{2n}, A^2x_{2n})d(SAx_{2n}, Bx_{2n-1}), d(A^2x_{2n}, Tx_{2n-1})d(Bx_{2n-1}, Tx_{2n-1})\} \\ & + c_3 d(SAx_{2n}, Bx_{2n-1})d(Tx_{2n-1}, A^2x_{2n}) \ .\end{aligned}$$

Letting n tend to infinity we have

$$[d(Az, z)]^2 \leq (c_1 + c_3)[d(Az, z)]^2$$

and since $c_1 + c_2 < 1$, it follows that $Az = z$.

Now since $A(X) \subseteq T(X)$, there exists a point v in X such that $Tv = Az = z$ Using inequality (3 2), we have

$$\begin{aligned}[d(A^2x_{2n}, Bv)]^2 \leq c_1 \max\{[d(SAx_{2n}, A^2x_{2n})]^2, [d(Tv, Bv)]^2, [d(SAx_{2n}, Tv)]^2\} \\ + c_2 \max\{d(SAx_{2n}, A^2x_{2n})d(SAx_{2n}, Bv), d(A^2x_{2n}, Tv)d(Bv, Tv)\} \\ + c_3 d(SAx_{2n}, Bv)d(Tv, A^2x_{2n}) .\end{aligned}$$

Letting n tend to infinity, we have

$$[d(Az, Bv)]^2 = [d(z, Bv)]^2 \leq c_1[d(z, Bv)]^2 ,$$

which implies that $z = Bv$ Since B and T are compatible and $Tv = Bv = z$, it follows from Proposition (2 1) that $TBv = BTv$ which implies that $Tz = Bz$ Similarly, applying inequality (3 2) to $[d(Ax_{2n}, Bz)]^2$, it follows that $Tz = Bz = z$

Since $B(X) \subseteq S(X)$, there exists point w in X such that $Sw = Bz = z$ Using inequality (3 2), we have

$$[d(Aw, z)]^2 = [d(Aw, Bz)]^2 \leq c_1[d(Sw, Aw)]^2 = c_1[d(z, Aw)]^2$$

and it follows that $z = Aw$ Since A and S are compatible and $Aw = Sw$, it follows from Proposition (2 1) that $ASw = SAw$ and so $Az = Sz$ Thus z is a common fixed point of A, B, S and T if A is continuous

The same result holds if we suppose that B is continuous instead of A

Finally, let us suppose that A and S have a second common fixed z' Then from inequality (3 2) we have

$$\begin{aligned}[d(z', z)]^2 = [d(Az', Bz)]^2 \leq c_1 \max\{[d(Sz', Az')]^2, [d(Tz, Bz)]^2, [d(Sz', Tz)]^2\} \\ + c_2 \max\{d(Sz', Az')d(Sz', Bz), d(Az', Tz)d(Bz, Tz)\} + c_3 d(Sz', Bz)d(Tz, Az') \\ = (c_1 + c_3)[d(z', z)]^2 ,\end{aligned}$$

and it follows that $z = z'$, since $c_1 + c_3 < 1$ Analogously, z is the unique common fixed point of B and T This completes the proof of the theorem.

We now prove the following common fixed point theorem for a compact metric space

THEOREM 3.2. Let (X, d) be a compact metric space and let A, B, S and T be continuous mappings of X into itself satisfying the conditions

$$A(X) \subseteq T(X) \quad \text{and} \quad B(X) \subseteq S(X) , \tag{3 5}$$

$$A \text{ and } B \text{ compatible with } S \text{ and } T \text{ respectively} \tag{3 6}$$

and the inequality

$$\begin{aligned}[d(Ax, By)]^2 < c \max\{[d(Sx, Ax)]^2, [d(Ty, By)]^2, [d(Sx, Ty)]^2\} \\ + \frac{1}{2}(1 - c)\max\{d(Sx, Ax)d(Sx, By), d(Ax, Ty)d(By, Ty)\} \\ + (1 - c)d(Sx, By)d(Ty, Ax)\end{aligned} \tag{3 7}$$

for all x, y in X for which the right hand side of (3.7) is positive, where $0 < c < 1$ Then A, B, S and T have a common fixed point z. Further, z is the unique common fixed point of A and S and of B and T

PROOF. Suppose first of all that there exists $c_1, c_2, c_3 \geq 0$, with $c_1 + 2c_2 < 1$ and $c_1 + c_3 < 1$ such that inequality (3.2) is satisfied. Then the result follows from Theorem 3 1.

Now suppose that no such c_1, c_2, c_3 exist. Then if $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are monotonically increasing sequences of real numbers converging to c, $\frac{1}{2}(1 - c)$ and $(1 - c)$ respectively, we can find sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned}[d(Ax_n, By_n)]^2 > a_n &\max\{[d(Sx_n, Ax_n)]^2, [d(Ty_n, By_n)]^2, [d(Sx_n, Ty_n)]^2\} \\ &+ b_n \max\{d(Sx_n, Ax_n)d(Sx_n, By_n), d(Ax_n, Ty_n)d(By_n, Ty_n)\} \\ &+ c_n d(Sx_n, By_n)d(Ty_n, Ax_n) \ , \end{aligned} \tag{3 8}$$

for $n = 1, 2, \ldots$ Since X is compact we may assume, by taking subsequences if necessary, that the sequences $\{x_n\}$ and $\{y_n\}$ converge to x and y respectively Letting n tend to infinity in inequality (3 8) we have, since A, B, S and T are continuous,

$$\begin{aligned}[d(Ax, By)]^2 \geq c &\max\{[d(Sx, Ax)]^2, [d(Ty, By)]^2, [d(Sx, Ty)]^2\} \\ &+ \frac{1}{2}(1-c)\max\{d(Sx, Ax)d(Sx, By), d(Ax, Ty)d(By, Ty)\} \\ &+ (1-c)d(Sx, By)d(Ty, Ax) \ . \end{aligned}$$

This is only possible if $Ax = By = Sx = Ty$ Since A and S are compatible and $Ax = Sx$, it follows from Proposition (2 1) that $ASx = A^2x = SAx$

Suppose that $A^2x \neq By$ Then using inequality (3 7) we have

$$\begin{aligned}[d(A^2x, By)]^2 &< c\max\{[d(SAx, A^2x)]^2, [d(Ty, By)]^2, [d(SAx, Ty)]^2\} \\ &\quad + \frac{1}{2}(1-c)\max\{d(SAx, A^2x)d(SAx, By), d(A^2x, Ty)d(By, Ty)\} \\ &\quad + (1-c)d(SAx, By)d(Ty, A^2x) \\ &= [d(A^2x, By)]^2 \ , \end{aligned}$$

a contradiction and so $A^2x = By = ABy$. It follows that $By = z$ is a fixed point of A Further,

$$Sz = SBy = SAx = ASx = ABy = Az = z$$

and it follows that z is a common fixed point of S and A

Similarly, we can prove that B and T have a common fixed point v If $z \neq v$, then on using inequality (3.7) we have

$$\begin{aligned}[d(z, v)]^2 &= [d(Az, Bv)]^2 \\ &< c\max\{[d(Sz, Az)]^2, [d(Tv, Bv)]^2, [d(Sz, Tv)]^2\} \\ &\quad + \frac{1}{2}(1-c)\max\{d(Sz, Az)d(Sz, Bv), d(Az, Tv)d(Bv, Tv)\} \\ &\quad + (1-c)d(Sx, Bv)d(Tv, Az) \\ &= [d(z, v)]^2 \end{aligned}$$

a contraction. Thus z is a common fixed point of A, B, S and T. It follows easily that z is the unique fixed point of A and S and of B and T.

Finally, note that some obvious corollaries can be obtained from Theorems 3 1 and 3.2 by letting $A = B$ and $S = T$.

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