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ON THE SURJECTIVITY OF LINEAR TRANSFORMATIONS

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ABSTRACT. Let B be a reflexive Banach space, X a locally convex space and $T : B \to X$ (not necessarily bounded) linear transformation A necessary and sufficient condition is obtained so that for a given $v \in X$ there is a solution for the equation $Tu = v$ This result is used to discuss the existence of an L^p-weak solution of $Du = v$ where D is a differential operator with smooth coefficients and $v \in L^p$

KEY WORDS AND PHRASES: Admissible linear operators, L^p-functions, harmonic functions
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1. INTRODUCTION

Let T be a (not necessarily bounded) linear operator from a reflexive Banach space B into a locally convex space X We obtain a necessary and sufficient condition for the existence of a solution $u \in B$ to the equation $Tu = v$, when $v \in X$ is known

In this context the following question arises naturally Let Ω be an open set in $\mathbb{R}^n$ and D a differential operator of order m with c^m-coefficients in Ω Given $v \in L^p(\Omega)$ does $Du = v$ have a weak solution $u \in L^p(\Omega)$?

When $p = 2$, Ω is a bounded domain and D has constant coefficients, L Hörmander (to see Corollary 1 14, M Schechter [1]) has proved that $Du = v$ has always a weak solution The proof depends heavily on Hilbert space techniques as applied to $L^2(\Omega)$ Our investigation here is around the form the above result of Hormander takes when only Banach space methods are available as in $L^p(\Omega)$

2. ADMISSIBLE LINEAR OPERATORS

Let B be a Banach space and X be a locally convex space. Let B' and X' denote the algebraic duals of B and X, B^* and X^* denote their topological duals.

Given $T : B \to X$, a linear operator not necessarily bounded, define the linear operator $T^* : X' \to B'$ as follows For $f \in X'$ and $x \in B$, $T^*f(x) = \langle x, T^*f \rangle = \langle Tx, f \rangle$

LEMMA 1. Let B be a reflexive Banach space and X be a locally convex space $T : B \to X$ is a linear operator, not necessarily bounded Suppose that there exists a subspace $H \subset X'$ such that $T^*(H) \subset B^*$ Then given $v \in X$, there exists $u \in B$, $\|u\| \le c$ such that $\langle Tu, f \rangle = \langle v, f \rangle$ for every $f \in H$ if and only if $|\langle v, f \rangle| \le c\|T^*f\|$

PROOF. Let $\langle Tu, f \rangle = \langle v, f \rangle$ with $\|u\| \le c$ and $f \in H$ Then $|\langle v, f \rangle| = |\langle u, T^*f \rangle| \le \|u\|$ $\|T^*f\| \le c\|T^*f\|$

Conversely, define the linear functional S on the subspace $T^*(H)$ so that, for $g \in T^*(H)$, $Sg = \langle v, f \rangle$ where $g = T^*f$ for some $f \in H$

S is well-defined, for, if $g = T^*f_1$, for some other $f_1 \in H$, then $|\langle v, f \rangle - \langle v, f_1 \rangle| = |\langle v, f - f_1 \rangle| \le c\|T^*(f - f_1)\| = 0$

It is clear that S is a bounded linear functional on the subspace $T^*(H) \subset B^*$ with $\|S\| \leq c$ and hence by Hahn-Banach theorem extends as a bounded linear functional on B^*, preserving the norm

This implies, since B is reflexive, that there exists $u \in B$ such that for every $h \in B^*$, $\langle u, h\rangle = Sh$ and $\|u\| = \|S\| = c$

In particular, if $h = T^*f$, $f \in H$, we have $\langle u, T^*f\rangle = S(T^*f) = \langle v, f\rangle$

Thus, for any $f \in H$, $\langle v, f\rangle = \langle u, T^*f\rangle = \langle Tu, f\rangle$

This completes the proof of the lemma

REMARK 2.1. The above lemma is inspired from section 1 6 of M Schechter [1] where the existence of a weak solution of a differential operator in the Hilbert space $L^2(\Omega)$ is investigated

DEFINITION 2.1. Let B be a Banach space and X be a locally convex space A linear operator $T : B \to X$, is said to be admissible if there exists a weak* dense subspace $M \subset X^*$ such that $T^*(M) \subset B^*$

PROPOSITION 2.1. Let B be a Banach space and X be a Fréchet space Let $T : B \to X$ be a linear operator Then T is continuous if and only if T is admissible

PROOF. If T is continuous, then for any $f \in X^*$ clearly $T^*f \in B^*$ and hence T is admissible

Conversely, let T be admissible with $T^*(M) \subset B^*$ where M is a weak*-dense subspace of X^* We will prove that T is continuous by showing that T is closed (W Rudin [2], p 50)

Let $x_n \in B$ be a sequence such that $\lim_n x_n = x$ and $\lim_n Tx_n = y$ Then for any $f \in M$, $\langle x_n, T^*f\rangle = \langle Tx_n, f\rangle$

Taking limits $\langle x, T^*f\rangle = \langle y, f\rangle$ which implies that $\langle Tx, f\rangle = \langle y, f\rangle$ for every $f \in M$ and consequently $\langle Tx, h\rangle = \langle y, h\rangle$ for every $h \in X^*$, since M is W^*-dense in X^*.

This implies that $Tx = y$ since X^* separates X, that is, T is closed

THEOREM 2.1. Let B be a reflexive Banach space and X be a locally convex space Let $T : B \to X$ be an admissible linear operator with $T^*(M) \subset B^*$ Then, for any given $v \in X$ there exists $u \in B$ such that $\|u\| \leq c$ and $Tu = v$ if and only if $|\langle v, f\rangle| \leq c\|T^*f\|$ for every $f \in M$

PROOF. In view of Lemma 1 (where we take $H = M$), it is enough to prove that the condition $\langle Tu, f\rangle = \langle v, f\rangle$ for every $f \in M$ is equivalent to the fact that $Tu = v$

Now, the condition above is equivalent to the fact $\langle Tu, h\rangle = \langle v, h\rangle$ for every $h \in X^*$, since M is dense in X^* with its W^*-topology

Since X^* separates points on the locally convex space X, the latter condition is equivalent to the fact $Tu = v$

3. WEAK SOLUTIONS IN $L^P(\Omega)$

Let Ω be a domain in $\mathbb{R}^n, n \geq 1$. Let $A = \sum_{|k|\leq m} a_k(x)D^k$ be a differential operator of order m, with $a_k(x) \in c^m(\Omega)$ Let A^* denote the adjoint operator Let $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$

THEOREM 3.1. With the above assumptions on A and p, let $f \in L^p(\Omega)$ be given Then there exists a weak solution of $Au = f$, $u \in L^p(\Omega)$ and $\|u\|_p \leq c$ if and only if $|\langle \phi, f\rangle| \leq c\|A^*\phi\|_q$ for all $\phi \in C_0^\infty(\Omega)$

PROOF. Suppose $f \in L^p$ and $Au = f$ has a weak solution $u \in L^p$, $\|u\| \leq c$

Define, for $\phi \in c_0^\infty(\Omega)$ and $g \in L^p(\Omega)$, $\langle \phi, g\rangle = \int_\Omega \overline{g}(x)\phi(x)dx$

Then, $|\langle \phi, f\rangle| = |\langle \phi, Au\rangle| = |\langle A^*\phi, u\rangle| \leq \|u\|_p \|A^*\phi\|_q \leq c\|A^*\phi\|_q$

Conversely, define the linear functional S on the subspace $A^*(c_0^\infty(\Omega))$ such that $S(A^*\phi) = \langle \phi, f\rangle = \int_\Omega \overline{f}\phi dx$

Then, as in Lemma 1, S is a well-defined linear functional on $A^*(c_0^\infty(\Omega)) \subset L^q(\Omega)$ with $\|S\| \leq c$ and hence extends as a continuous linear functional on $L^q(\Omega)$, so that there exists $u \in L^p(\Omega)$ satisfying the condition $S(v) = \langle v, u\rangle$ for all $v \in A^*(c_0^\infty(\Omega))$ and $\|u\|_p = \|S\| \leq c$

In particular, for any $\phi \in c_0^\infty(\Omega)$, $\langle \phi, f\rangle = S(A^*\phi) = \langle A^*\phi, u\rangle = \langle \phi, Au\rangle$ Hence u is a weak solution of $Au = f$

THEOREM 3.2. Let $f \in L^1_{loc}(\Omega)$ Then there exists a bounded weak solution u of the equation $Au = f$ if and only if $\left|\int_\Omega \overline{f}(x)\phi(x)dx\right| \leq C\|A^*\phi\|_1$, for every $\phi \in c_0^\infty(\Omega)$

PROOF. In view of the above theorem, we will give here only a few details of the proof

On $A^*(c_0^\infty(\Omega))$, considered as a subspace of $L^1(\Omega)$, define the linear functional S such that $S(A^*\phi) = \langle \phi, f\rangle = \int \overline{f}\phi dx$ Then S extends as a bounded linear functional on $L^1(\Omega)$ so that there exists $u \in L^\infty(\Omega)$ such that $Sg = \langle g, u\rangle$ for every $g \in L^1(\Omega)$

This leads to the fact that u is a weak solution of $Au = f$

In the context of the above theorem where we were looking for a bounded weak solution of a differential equation, the following proposition concerning the bounded solutions of the Laplacian in $\mathbb{R}^n$ is of interest

PROPOSITION 3.1. Let $f \in C_0^\infty(\mathbb{R}^n)$, having compact support K, be given in $\mathbb{R}^n$ Then, if $n \geq 3$, there always exists a bounded $u \in c^\infty(\mathbb{R}^n)$ such that $\triangle u = f$, if $n = 1$ or 2, such a bounded c^∞-solution exists if and only if $\int_K f(x)dx = 0$

PROOF. Since $\triangle$ is an elliptic differential operator with constant coefficients, there always exists some $u \in c^\infty(\mathbb{R}^n)$ such that $\triangle u = f$ Here we are looking for a bounded function u in $c^\infty(\mathbb{R}^n)$

Let

$$E_n(x) = \begin{cases} |x| & \text{if } n = 1 \\ \log|x| & \text{if } n = 2 \\ -\dfrac{1}{|x|^{n-2}} & \text{if } n > 3 \end{cases}$$

Now, using the results in [2], we can show that for a fixed $y \in K$ and any $x \in K^c$,

$$u(x) = \left(\int f(x)dx\right)\beta_n E_n(x-y) + l(x)$$

where $l(x)$ is a bounded harmonic function in K^c if $n \geq 2$ (and affine bounded if $n = 1$)

Here $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{1}{2\pi}$ and $\beta_n = \frac{1}{(n-2)\alpha_n}$ if $n \geq 3$, α_n being the measure of the unit sphere in $\mathbb{R}^n$

Consequently, using the fact that $E_n(x)$ is bounded in a neighborhood of the point at infinity if and only if $n \geq 3$, we arrive at the conclusion of the proposition

NOTE. Since a bounded harmonic function outside a compact set in $\mathbb{R}^n$, $n \geq 2$, tends to a limit at infinity, if u is a bounded solution of $\triangle u = f \in c_0^\infty$, we can choose $u_0 \in c^\infty(\mathbb{R}^n)$ so that u_0 tends to 0 at infinity and satisfies the condition $\triangle u_0 = f$ In this case, u_0 is unique

4. SURJECTIVITY ON THE SOBOLEV SPACES

We conclude this article with a remark on the solutions of a differential operator on the Sobolev spaces $H^s(\mathbb{R}^n)$

We make use of the following properties.

i) For each real s, $H^s(\mathbb{R}^n)$ is a Hilbert space such that $H^s \subset H^t$ if $t \leq s$

ii) H^s is the completion of c_0^∞ in the norm $\|\cdot\|_{H^s}$

iii) For any s, H^{-s} represents the topological dual of H^s

iv) If $s > \frac{n}{2} + k$ where k is a nonnegative integer, then $H^s \subset c^k$

v) If A is a differential operator of order m with c^m-coefficients, $A^*(c_0^\infty) \subset L^2 \subset H^s$ for any $s \geq 0$

vi) If A is a differential operator of order m with c^∞-coefficients, $A^*(c_0^\infty) \subset H^s$ for every real s

Then, with arguments similar to those utilized to prove some of the earlier results, we obtain

THEOREM 4.1. Let T be a distribution in $\mathbb{R}^n$, $n \geq 1$ Suppose that A is a differential operator of order m satisfying one of the following two sets of assumptions

a) A has c^m-coefficients and $s \geq 0$

b) A has c^∞-coefficients and s is any real number

Then $T = Au$ in the sense of distribution, for some $u \in H^s$, if and only if $|T(\phi)| \leq c\|A^*\phi\|_{H^{-s}}$ for all $\phi \in c_0^\infty(\mathbb{R}^n)$

REMARK 4.1. Let A be a differential operator of order m with coefficients either constants or from the Schwartz's space (i e rapidly decreasing c^∞-functions) Then if $|T(\phi)| \leq c\|A^*\phi\|_{H^{-s}}$ for all $\phi \in c_0^\infty$, we have as in the above theorem, $T = Au, u \in H^s$

But, in this special case, $T \in H^{s-m}$ and consequently, if $s > \frac{n}{2} + m$, then $T = Au$ in the classical sense i e u is a strong solution of the differential equation

$H^k(\Omega)$-spaces Let now Ω be an open set in $\mathbb{R}^n, n \geq 1$ Recall that for any positive integer k, $H_0^k(\Omega)$ is defined as the closure of $c_0^\infty(\Omega)$ in $H^k(\Omega)$ For any $v \in L^2(\Omega)$, define $\|v\|_{-k} = \sup_{u \in H_0^k(\Omega)} \frac{|\langle v,u\rangle|}{\|u\|_{H^k(\Omega)}}$

Then, if $H^{-k}(\Omega)$ denotes the completion of $L^2(\Omega)$ in the norm $\|\cdot\|_{-k}$, $H^{-k}(\Omega)$ is the topological dual of $H_0^k(\Omega)$ for any integer $k \geq 0$ (see Al-Gwaiz [4], p 191)

With this background, we can state an analogue of Theorem 4 1 as follows

THEOREM 4.2. Let T be a distribution in an open set Ω in $\mathbb{R}^n$, $n \geq 1$ Suppose A is a linear differential operator with $c^\infty(\Omega)$-coefficients Then, for any integer $k \geq 0$, there exists $u \in H^{-k}(\Omega)$ such that $Au = T$ if and only if $|T\phi| \leq c\|A^*\phi\|_{H^k(\Omega)}$ for all $\phi \in c_0^\infty(\Omega)$

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