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EXOTIC STRUCTURES ON QUOTIENT SPACES OF S^3-ACTIONS

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ABSTRACT. A correct version of some results by A Rigas regarding S^3 actions on $S^7 \times S^3$ and on the symplectic group Sp_2 with quotients exotic seven-spheres is presented

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1. INTRODUCTION

The present note is a result of our interest in finding exotic structures on 7-dimensional manifolds (cf Guest and Micha [3], Astey, Micha and Pastor [1]) and its purpose is to correct some mistakes that occur in a paper by A Rigas [6] Our contribution is simply to provide the correct statement and a different proof of the key corollary that appears on page 76 of Rigas [6], but we take the opportunity to restate several results of the paper which refer to the existence of free S^3 actions on $S^7 \times S^3$ and on the symplectic group Sp_2 with quotients exotic seven-spheres, which also appear incorrectly stated in that paper

2. MAIN RESULTS

We begin by recalling some definitions and notation of Rigas [6] Principal S^3 bundles over S^4 are classified by $\pi_3 S^3$ which is naturally isomorphic to the group of integers Z Let P_n denote the total space of the bundle corresponding to the integer n Similarly, the principal S^3 bundles over S^7 are classified by $\pi_6 S^3$ We shall denote by E_i the total space of the bundle corresponding to $i \in \pi_6 S^3 \cong Z_{12}$ The isomorphism here is such that $E_1 \cong Sp_2$ Let $\tilde{P}_n$ denote the pull-back of P_n under the Hopf map $S^7 \to S^4$ Then, as a principal S^3 bundle, $\tilde{P}_n$ is classified by the composition

$$S^7 \xrightarrow{h} S^4 \xrightarrow{f_n} S^4 \to BS^3$$

where f_n denotes the map of degree n, and the rightmost arrow is the inclusion of the bottom cell

THEOREM. The bundles $\tilde{P}_n$ and $E_{n(n-1)/2}$ are isomorphic as principal S^3 bundles over S^7

This theorem is the correct version of the corollary on page 76 of Rigas [6] The mistake leading to the incorrect statement in Rigas [6] occurs in the calculation of the map $f_n \circ h$, where the author fails to iterate correctly a formula of Hilton [4]. An alternative proof using a different bundle decomposition is presented in §3 below.

It follows from the theorem that

(a) $\tilde{P}_n$ and the trivial bundle $S^7 \times S^3$ are isomorphic only if $n \equiv 0, 1, 9$ or $16 \bmod 24$.

(b) $\tilde{P}_n$ and the canonical bundle $Sp_2 \to S^7$ are isomorphic only if $n \equiv 2$ or $23 \bmod 24$.

In particular, $\tilde{P}_{13}$ is not a trivial bundle. This renders §4 of Rigas [6] invalid. The theorem also allows us to rectify the statements of two important results of Rigas [6] as follows.

COROLLARY. There exist free actions of S^3 on $S^7 \times S^3$ with quotient the exotic seven-spheres of Eells-Kuiper invariants 16, 40 and 48.

COROLLARY. There exist free actions of S^3 on Sp_2 with quotient the exotic seven-spheres of Eells-Kuiper invariants 2, 26, 34 and 42.

3. PROOF OF THE THEOREM

As is shown in Rigas [6], S^7 can be decomposed into two solid tori $U \cong S^3 \times D^4$ and $V \cong D^4 \times S^3$ such that the restriction of the bundle $\tilde{P}_n$ to each torus is trivial. Moreover, the transition map

$$\lambda_{UV} : S^3 \times S^3 \to S^3$$

is given by

$$\lambda_{UV}(x, y) = x^{n-1}(yx^{-1})^{n-1}y^{-(n-1)},$$

where the group structure of unit quaternions is understood on S^3. Since the commutator $xyx^{-1}y^{-1}$ generates $\pi_6 S^3$ (Hilton and Roitberg [5]) and since λ factors through S^6, the theorem is a consequence of the following result.

PROPOSITION. The map $\lambda : S^3 \times S^3 \to S^3$ given by $\lambda(x,y) = x^{n-1}(yx^{-1})^{n-1}y^{-(n-1)}$ is homotopic to $(xyx^{-1}y^{-1})^{n(n-1)/2}$.

We first prove the following lemma.

LEMMA. The maps $x^k y^l x^{-k} y^{-l}$ and $(xyx^{-1}y^{-1})^{kl}$ are homotopic.

PROOF. Consider the following commutative diagram

$$\begin{tikzcd} S^3 \times S^3 \arrow[r, "\alpha"] \arrow[d, "p"] & S^3 \times S^3 \arrow[r, "\beta"] \arrow[d, "p"] & S^3 \arrow[r, "\gamma"] & S^3 \\ S^6 \arrow[r, "f_{kl}"'] & S^6 \arrow[ur, "\omega"'] & & \end{tikzcd}$$

where $\alpha(x, y) = (x^k, y^l)$, $\beta(x, y) = xyx^{-1}y^{-1}$, $\gamma(x) = x^{kl}$, p is the projection that collapses the 3-skeleton, f_{kl} is a map of degree kl, and ω is the generator of $\pi_6 S^3$. But since S^3 is an H-space, homotopy compositions are biadditive (Whitehead [7], p. 479), so $\omega \circ f_{kl} \simeq \gamma \circ \omega$. Therefore,

$$x^k y^l x^{-k} y^{-l} = \beta \circ \alpha \simeq \gamma \circ \beta = (xyx^{-1}y^{-1})^{kl}$$

We now prove the proposition by induction on n. Let $c = xyx^{-1}y^{-1}$. If we take $k = 1$ and $l = -1$ in the lemma we obtain $xy^{-1}x^{-1}y \simeq c^{-1} = yxy^{-1}x^{-1}$. Hence,

$$\begin{aligned} c^{-1}ycy^{-1} &= (yxy^{-1}x^{-1})ycy^{-1} \\ &= y(xy^{-1}x^{-1}y)cy^{-1} \\ &\simeq yc^{-1}cy^{-1} \\ &= 1, \end{aligned}$$

that is, $cy^{-1} = y^{-1}c$.

Assume now that $x^n(yx^{-1})^n y^{-n} = c^{k(n)}$. Clearly, $k(1) = 1$. But now

$$\begin{aligned}
x^n(yx^{-1})^n y^{-n} &= x^n yx^{-1}(yx^{-1})^{n-1}y^{-n} \\
&= (x^n yx^{-n}y^{-1})yx^{n-1}(yx^{-1})^{n-1}y^{-n} \\
&= c^n y(x^{n-1}(yx^{-1})^{n-1}y^{-(n-1)})y^{-1} \\
&= c^n yc^{k(n-1)}y^{-1} \\
&= c^{n+k(n-1)}.
\end{aligned}$$

Therefore, $k(n) = n + k(n-1)$, that is, $k(n) = n(n+1)/2$ This proves the proposition

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